2.0 SDOF Lumped Parameter Dynamics

2.1 The Fourier Series and Fourier Integral

2.1.1 Forms of the Fourier Series

Use of the Fourier Series (FS) is a type of curve-fitting exercise in which a function is represented by the sum of a cosine plus sine term. That is, $p(t) = (a_1 \cos \omega t + b_1 \sin \omega t) + (a_2 \cos 2\omega t + b_2 \sin 2\omega t) +$ ……. Given p(t), the aim is to determine the constants a_1, b_1, a_2, b_2, ….. The FS can be expressed in various forms that can simplify subsequent calculations. In structural dynamics p(t) is the dynamic loading, also called the forcing function, on a structure. The FS is particularly useful for representing periodic loading.

2.1.1.1 Trigonometric Form

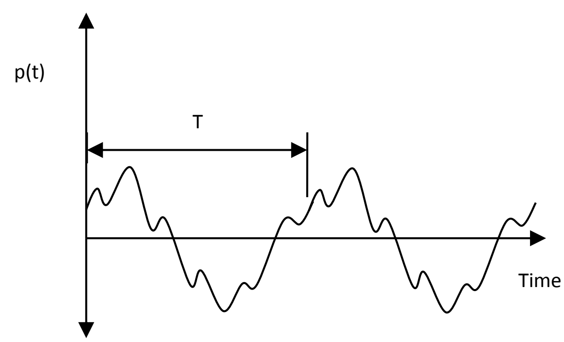

Figure 5

Consider any periodic function of time such as that shown in Figure 5, where T is the period.
p(t) expressed as a Fourier Series (FS) is given by,

$$p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \quad (2.1)$$

a_0, a_n, and b_n are real constants, t is time, and $\omega = 2\pi/T$, is the frequency. a_0, a_n, and b_n are called the real Fourier coefficients.

a_n and b_n are respectively equivalent to the horizontal and vertical projections of the radius of a circle in which the length or magnitude of the radius is A_n. Therefore, if ϕ_n is the angle between A_n and a_n, then,

$$\tan \phi_n = b_n/a_n \quad (2.2)$$

ϕ_n is therefore the phase difference, lag, or angle between a_n and b_n.

Also,

$$A_n = \sqrt{a_n^2 + b_n^2} \quad (2.3)$$

Eqn (2.1) is rewritten as,

$$p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \left(\frac{a_n}{A_n} \cos n\omega t + \frac{b_n}{A_n} \sin n\omega t\right) \tag{2.4}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \left(\cos\phi_n \cos n\omega t + \sin\phi_n \sin n\omega t\right)$$

Hence, if $A_0 = a_0/2$, and $\phi_0 = 0$,

$$p(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \phi_n) \tag{2.5}$$

Plots of A_n vs frequency, and ϕ_n vs frequency, are called the amplitude or magnitude spectrum, and the phase spectrum, respectively. They are discrete and occur at the Fourier frequencies $n\omega = 2n\pi/T$.

Note that A_0 is called the "DC" (i.e. direct current) term, and is the average value of p(t). This enables its value to be determined by visual inspection in many cases. Eqn (2.5) is called the real sinusoid form of the FS.

The Fourier Coefficients are determined from,

$$a_0 = \frac{1}{T}\int_{-T}^{T} p(t)dt \tag{2.6}$$

$$a_n = \frac{1}{T}\int_{-T}^{T} p(t)\cos\left(\frac{2n\pi t}{T}\right)dt \tag{2.7}$$

$$b_n = \frac{1}{T}\int_{-T}^{T} p(t)\sin\left(\frac{2n\pi t}{T}\right)dt \tag{2.8}$$

Hence knowing p(t) in closed-form, a_0, a_n, and b_n can be determined as formulae, and back-substituted in eqn (2.1) or (2.5).

If p(t) is an odd function, then $a_n = 0$, and if it is an even function, then $b_n = 0$. A function is odd if f(-x) = -f(x). A function is even if f(-x) = f(x). Graphically, for an even function the y-values for the -x values are a mirror image of the y-values for the +x values. For an odd function the y-values for the -x values are a mirror image of the y-values for the –x values, but flipped about the x-axis.

2.1.1.2 Exponential Form

The exponential form of the FS is due to the Euler's formula for a complex number which leads to,

$$\cos n\omega t = \frac{1}{2}(e^{in\omega t} + e^{-in\omega t})$$

$$\sin n\omega t = \frac{-i}{2}(e^{in\omega t} - e^{-in\omega t})$$

Substituting in eqn (2.1) and simplifying,

$$p(t) = \sum_{n=-\infty}^{\infty} P_n\, e^{in\omega t} \tag{2.9}$$

where $P_0 = a_0/2$ (2.10)

$$P_n = \frac{a_n - ib_n}{2} \quad (2.11)$$

$$P_{-n} = \frac{a_n + ib_n}{2} \quad (2.12)$$

Eqn (2.9) is the exponential form of the FS; $i = \sqrt{(-1)}$. Note that in eqn (2.9) the limits are from -∞ and not 1 as in eqn (2.1).

P_n and P_{-n} are complex numbers and P_{-n} is the complex conjugate of P_n. Eqn (2.10) applies when n = 0. Eqn (2.11) applies when n > 0, and eqn (2.12) applied when n < 0, in which case the Fourier frequencies are negative. P_n is calculated from,

$$P_n = \frac{1}{T}\int_t^{t+T} p(t)e^{-in\omega t}dt \quad (2.13)$$

$|P_n|$ is the amplitude, magnitude, or modulus of P_n. The amplitude spectrum of the exponential form of the FS is the graph of $|P_n|$ vs frequency, and the frequencies have a negative range. This is called a two-sided spectrum. Like the amplitude spectrum of the trigonometric FS, the spectrum is a set of discrete values. The concept of negative frequencies is counter-intuitive so a one-sided spectrum is frequently used. For a one-sided spectrum the values are twice those of a two-sided spectrum.

The spectral values of the trigonometric form of the FS are related to those of the exponential form of the FS by the following, assuming one-sided spectra (i.e. positive frequencies only). This is indicated by eqn (2.3).

$$|2P_n| = \sqrt{a_n^2 + b_n^2} \quad (2.14)$$

$$\tan \phi_n = Im(P_n)/Re(P_n) \quad (2.15)$$

where Im is the imaginary part, and Re is the real part.

Other useful relationships, which hold for any complex number, are,

$$Re\,(P_n\, e^{in\omega t}) = ½\,(P_n\, e^{in\omega t} + P_n^*\, e^{-in\omega t}) \quad (2.16)$$

$$P_n P_n^* = |P_n|^2 \quad (2.17)$$

From the above, it is therefore important to be mindful of which type of FS is required, or is referred to.

2.1.2 The Fourier Integral

The Fourier Integral (FI) is particularly useful for representing non-periodic dynamic loading, also called arbitrary loading. Arbitrary loading includes the cases of short-term loading (also called transient loading e.g. blast loading), and random loading. Therefore the FI is of wide application in structural dynamics particularly for solution in the frequency domain.

Recall Figure 5. This suggests that it may be possible to use the FS for non-periodic loading as the loading can be non-periodic in T. However, the FS is periodic within any range of t and therefore repeats when t > T. For non-periodic loading the structural response after the loading has stopped is typically of interest therefore response calculations for t > T will be inaccurate. Hence the FS cannot be used to represent non-periodic loading.

Recall eqn (2.9). The frequency corresponding to each term is $n\omega = 2n\pi/T$ and in the spectral representation, the discrete lines are separated by a finite frequency interval. Hence as the frequency interval tends to zero, in the limit the summation becomes an integral and eqn (2.9) becomes,

$$p(t) = \int_{-\infty}^{\infty} P(f)e^{i2\pi ft}df \tag{2.18}$$

where,

$$P(f) = \int_{-\infty}^{\infty} p(t)e^{-i2\pi ft}dt \tag{2.19}$$

Note that in these equations the measure of frequency is f, the rectilinear frequency (Hz or cps), and not ω, the circular frequency (rps). Note also that P(f) and p(t) are continuous functions.

P(f) is called the Fourier Transform (FT) of p(t), and p(t) is called the Inverse Fourier Transform (IFT) of P(f). They are referred to as a transform pair.

In this text P(f) is called the direct Fourier Transform since it can be obtained in closed-form if p(t) is expressed as a function. Similarly, the Fourier coefficients of the FS can also be obtained in closed-form. However, in many cases (e.g. data available as load, time points), the Fourier Transform cannot be obtained in closed-form, and a numerical approach is required. This is called a Discrete Fourier Transform (DFT) which under certain conditions can be optimized in which case it is called the Fast Fourier Transform (FFT). The FT is a powerful mathematical operator in general and readily provides the response output for a given input, for linear problems.

2.2 Solutions

2.2.1 Introduction – The Atomic Solutions

Atoms are the building blocks of elements, molecules, compounds, etc, hence of all matter. In the same way, for linear problems, the solutions for the case of General Harmonic loading, and Impulse loading, are the building blocks for determining the solutions for all other types of loading.

This is because of the Principle of Superposition which simply means, for example, that if you know the solution for "x", then to get the solution for "2x", just multiply the solution for "x", by 2.

General Harmonic loading is given by A cos ωt + B sin ωt and its solution is as presented below in section 2.2.2.1.1. However, as shown above in section 2.1.1, any load represented as a Fourier Series (i.e. any type of periodic loading) is a sum of terms of the form A cos ωt + B sin ωt, so knowing the solution for General Harmonic loading, the solution for any periodic loading is obtained merely by suitably multiplying or superimposing, the solution for the General Harmonic loading.

Likewise, for any non-periodic or arbitrary loading, at any point in time, t, that load can be considered an impulse load. Therefore any loading can be considered as a set of impulses. Hence by knowing the solution for an impulse load, as presented in section 2.2.2.1.2 below, to get the solution for an arbitrary load, the result for the impulse load is multiplied or superimposed, for each load value at time, t.

In the following sections, solutions are presented in both the time and the frequency domains. This means as functions of time, and as functions of frequency, respectively. Solutions in the time domain describe when any item of interest occurs, such as a peak response, but it does not easily describe what aspects of the loading are most influential. A frequency domain solution however, gives only peak responses but not when they occur. It however does indicate which aspects of the loading are most influential by revealing their frequency content. For problems that are sufficiently simple that closed-form solutions are readily available, the frequency domain approach gives rapid results and since civil engineers are very concerned about peak values, the frequency domain approach may be preferred.

Practical problems are typically sufficiently complicated that closed-form solutions are not available, and since numerical methods must then be used, the two approaches tend to have approximately equal advantages as disadvantages. The frequency domain approach is generally preferred for random loading, due to the possibility of linearization.

Note that since these "atomic solutions" are used with the Principle of Superposition, this "atomic solutions" approach is only valid for linear systems. However, for very many problems of interest in civil engineering, a non-linear problem can be converted to an equivalent linear one, thus making the linear solutions applicable. The non-linear case is presented below in section 2.2.3.

2.2.2 Linear

2.2.2.1 Time Domain

2.2.2.1.1 General Harmonic

A SDOF system is simply a representation of the structure as a single lumped-mass that can only move along one line, but possibly forward or backward. This single way the structure can move is hence called a single degree of freedom (SDOF).

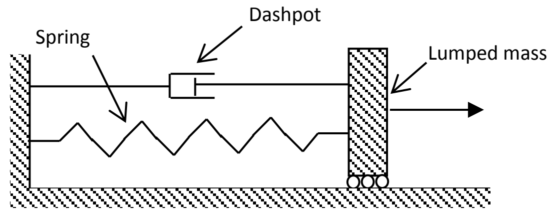

Figure 6

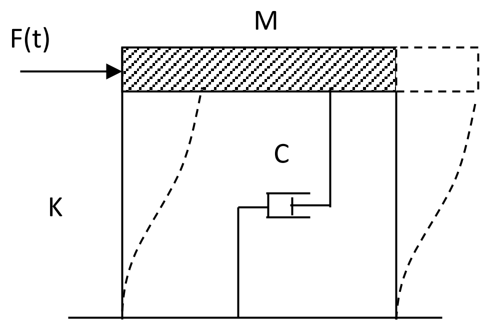

Figure 7(a)

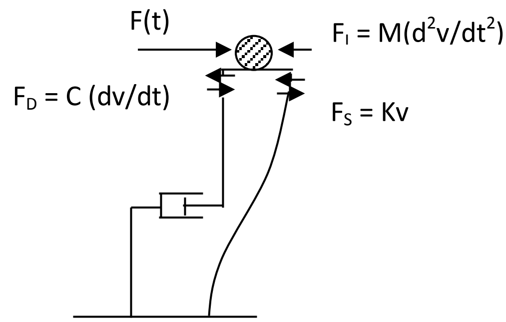

Figure 7(b)

Typical structural models for SDOF systems are as shown in Figures 6 and 7(a) above. In each case only the lumped mass is considered to have mass. Also, the essential elements of the model are: the lumped mass, the spring and the dashpot. The spring element represents the "springyness" of the structure, and the dashpot element represents anything that may oppose the motion of the structure such as internal friction. Both Figure 6 and 7(a) are the initial idealizations of the structure. Civil engineers can usually identify more easily with the structural model as Figure 7(a) which is a simple portal frame with the mass at the beam level. Therefore this model will be used throughout this text.

Figure 7(b) shows the free body diagram (FBD) associated with the structural model and makes it easier to identify the relevant forces on the system and hence formulate the equation of motion.

With the applied dynamic force, also called the forcing function or excitation, as F(t), the inertia force is $F_i(t)$, the spring force is $F_s(t)$, and the damping force is $F_D(t)$. Note that the dynamic force F(t) is directly applied to the mass.

According to d'Alembert's principle of dynamic equilibrium, but examining the forces directly acting on the mass,

$$F_I + F_D + F_S = F(t) \qquad (2.20)$$

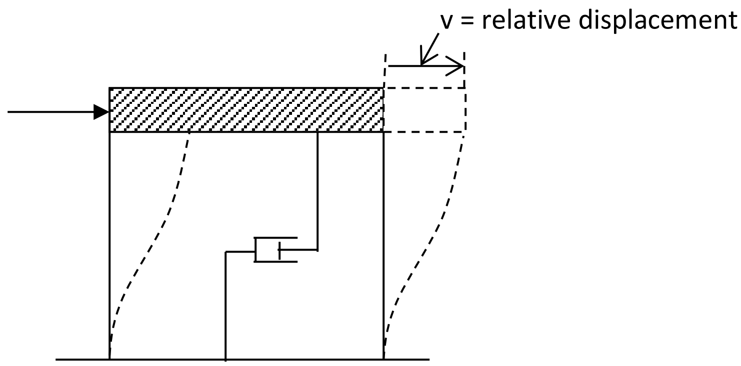

Figure 8

The displacement of the structure relative to its base is defined as v, as shown in Figure 8.

Hence, from Newton's Second Law,

$F_I(t) = M(d^2v/dt^2)$

In classical damping, also called viscous damping, it is assumed that the damping force is proportional to the velocity of the mass. This is the typical type of damping considered in structural dynamics and other types can frequently be converted to equivalent viscous damping. Hence, considering the viscous damping constant as C,

$F_D(t) = C\ (dv/dt)$

From elementary structural mechanics the spring force is,

$F_S(t) = Kv(t)$

K is a constant in the above eqn hence the term is linear with respect to v. K is the lateral stiffness of the structure. For a nonlinear system K is not a constant but is a function of v (i.e. K(v)).

Substituting in eqn (2.20) and dropping the (t) from v for simplification,

$$M(d^2v/dt^2) + C(dv/dt) + Kv = F(t) \quad (2.21)$$

From the theory of ordinary differential equations (ODEs), eqn (2.21) is a linear second-order non-homogenous ODE with constant coefficients. Hence its solution is given by,

$$v = v_c + v_p \quad (2.22)$$

where v_c is the complementary function and v_p is the particular solution. The complementary function is the solution of the homogeneous from of eqn (2.21) in which case the RHS is zero. When the RHS is zero there is no forcing function so this is called free vibration. Therefore, the solution of the free vibration problem is part of the solution of the total response given by eqn (2.22). The cases of undamped and damped free vibration are therefore presented next.

Undamped Free Vibration

For the case of free vibration there is no applied load so the RHS of (2.21) is zero. In the most general case, the mass can have a displacement value at time $t = 0$ hence $v(0) \neq 0$. If the system has no damping then the second term on the LHS of eqn (2.21) is zero. This case is called undamped free vibration and is both instructive and useful. The presence of damping has important consequences depending on the amount of damping present and this is best seen by comparison with the undamped case. More significantly, the solution to the undamped free vibration problem is essential in the study of the multiple-degree-of-freedom case (MDOF), presented in section 3. In that case it is required for calculating the mode shapes of the structure.

Therefore for undamped free vibration eqn (2.21) becomes,

$$M(d^2v/dt^2) + Kv = 0 \qquad (2.23)$$

As eqn (2.23) is second-order, its general solution has two constants. The general harmonic relation satisfies eqn (2.23). Therefore,

$$v(t) = A \cos \omega_n t + B \sin \omega_n t \qquad (2.24)$$

ω_n is called the *natural frequency* of the system. By substitution into eqn (2.23) it can be shown that $\omega_n = \sqrt{(K/M)}$, hence $T = 2\pi\sqrt{(M/K)}$. Also, consideration of the initial conditions gives the constants A and B as,
$A = v(0)$
$B = (dv/dt)_{t=0}/\omega_n$

From the trig identity $A\cos \theta + B\sin \theta = X \cos(\theta - \alpha)$, $X = (A^2 + B^2)^{12}$, and $\tan \alpha = (B/A)$. Graphically, eqn (2.24), the displacement response for undamped free vibration, is as shown below.

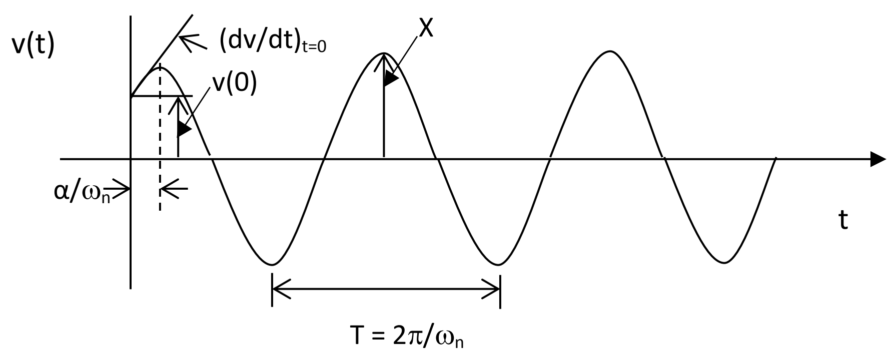

Figure 9

Damped Free Vibration

If viscous damping is present in the system, (2.23) becomes,

$$M(d^2v/dt^2) + C (dv/dt) + Kv = 0 \qquad (2.25)$$

Dividing by M, we get,

$$d^2v/dt^2 + 2\xi\omega_n(dv/dt) + \omega^2 v = 0$$

where $2\xi\omega_n = C/M$ (ξ is pronounced "zeta")

The solution of is,

$v = A \exp(\lambda_1 t) + B \exp(\lambda_2 t)$ (2.26)

where $\lambda_1, \lambda_2 = \omega_n[- \xi \pm (\xi^2 - 1)^{1/2}]$ (2.27)

(2.27) indicates that the solution changes form according to the value of ξ^2.

If $\xi^2 < 1$,

$v = X \exp(-\xi\omega_n t) \sin(\omega_D t + \theta)$ (2.28)

where $X = (A^2 + B^2)^{1/2}$ $\theta = \tan^{-1} B/A$

$\omega_D = (1 - \xi^2)^{1/2} \omega_n$ (2.29)

For initial conditions of v = 0 and $(dv/dt)_{t=0}$, A and B can be shown to be,

$A = v(0)$
$B = [(dv/dt)_{t=0} + \xi\omega_n v(0)]/ \omega_D$

ω_D is called that the *damped vibration circular frequency*. The portion of the equation before the sine of (2.28) and (2.30), indicates that the system experiences a decaying oscillation with t, and is an envelope of the response.

If $\xi^2 > 1$, the system does not oscillate since the effect of the damping overshadows the oscillation. This is called *overdamped* system. If $\xi^2 < 1$, this is called an *underdamped* system. Vibration can only occur if the system is underdamped.

The condition $\xi^2 = 1$ indicates a limiting value of damping at which the system loses its vibratory characteristics; this is called *critical damping*. From (2.25), the critical damping constant,

$C_{cr} = 2\omega nM = 2(MK)^{1/2}$ (2.30)

ξ is defined in terms of C_{cr} as, $\xi = C/C_{cr}$. Hence ξ is called the *fraction of critical damping,* or the *damping ratio*. Typical values of ξ are 5% for reinforced concrete, and 2% for steel.

From (2.28) we get a means for experimentally determining ξ. It can be shown that for successive vibration amplitudes of a damped system,

$\ln(v_n/v_{n+1}) \approx 2\pi\xi$. This is called the *logarithmic decrement.*

Graphically, eqn (2.28), the displacement response for damped free vibration, is as shown below.

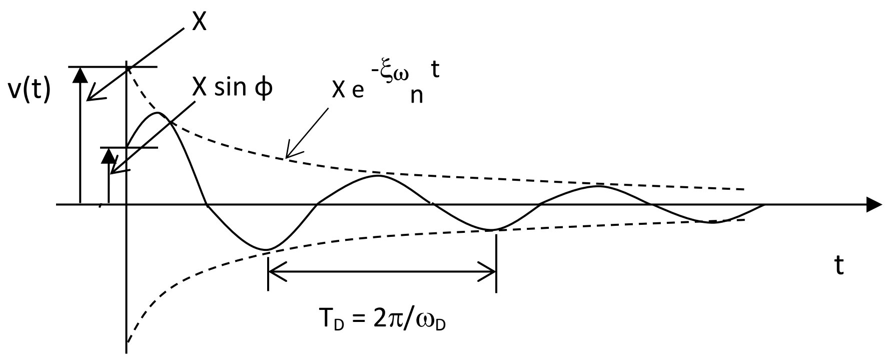

Figure 10

Note that eqn (2.28) can also be written as,

$$v(t) = e^{-\xi\omega_n t}\left(v(0)\cos\omega_d t + \frac{\left(\frac{dv}{dt}\right)_{t=0} + \xi\omega_n v(0)}{\omega_d}\sin\omega_d t \right) \tag{2.31}$$

General Harmonic Loading

Returning to equations (2.21) and (2.22), general harmonic loading is when the forcing function or dynamic loading $F(t) = p_o \sin \omega' t$, or $p_o \cos \omega' t$. Consider,

$$M(d^2v/dt^2) + C(dv/dt) + Kv = F(t) = p_o \sin \omega' t \tag{2.32}$$

In eqn (2.22) the complementary function is the solution of the homogeneous equation which is the solution of the free vibration problem given by (2.28) or (2.31). Because of damping v_c dies off quickly so is called the transient response. The particular solution is called the steady-state solution because after v_c dies off, the vibration of the system continues under the action of the forcing function. It also has the same frequency as the forcing function but lags behind it. This means that the peak response of the system occurs some time after the peak force of the forcing function. Therefore, due to the trigonometric identities, the steady-state response is of the form $A \cos \omega' t + B \sin \omega' t$.

$$v = v_p = A\cos\omega' t + B\sin\omega' t \tag{2.33}$$

Dividing (2.32) by M, then substituting (2.33),

$$-B\omega'^2 \sin\omega' t - A\omega'^2 \cos\omega' t + \frac{C}{M}\omega' B\cos\omega' t - \frac{C}{M}\omega' A\sin\omega' t + \frac{K}{M}B\sin\omega' t + \frac{K}{M}A\cos\omega' t = \frac{p_0}{M}\sin\omega' t \tag{2.34}$$

Simplifying,

$$\left(\frac{K}{M} - \omega'^2\right)B - \frac{C}{M}\omega' A = \frac{p_0}{M} \tag{2.35}$$

and,

$$\frac{C}{M}\omega' B + \left(\frac{K}{M} - \omega'^2\right) A = 0 \tag{2.36}$$

Solving (2.35) and (2.36) for A and B,

$$A = \frac{-(C/M)\omega'(p_0/M)}{\left[(K/M) - \omega'^2\right]^2 + \left[(C/M)\omega'\right]^2} \tag{2.37}$$

$$B = \frac{\left[(K/M) - \omega'^2\right](p_0/M)}{\left[(K/M) - \omega'^2\right]^2 + \left[(C/M)\omega'\right]^2} \tag{2.38}$$

Let r = ω' / ω_n . This is called the frequency ratio.

Also, let X_0 = equivalent static deflection = p_0/K. Hence,

$$\frac{p_0}{M} = \frac{p_0}{M}\frac{M}{K}\omega_n{}^2 = \frac{p_0}{K}\omega_n{}^2$$

Therefore, (2.37) and (2.38) become,

$$A = \frac{-2\xi r X_0}{(1-r^2)^2 + (2\xi r)^2} \tag{2.39}$$

$$B = \frac{(1-r^2)X_0}{(1-r^2)^2 + (2\xi r)^2} \tag{2.40}$$

Amplitude and Phase Angle Form:-

Alternatively, eqn (2.33) can be written as,

$$v_p = X_f \sin(\omega' t - \phi) \tag{2.41}$$

The subscript “f” refers to “forcing”, as opposed to the “o” referring to the static condition.

Hence,

$$X_f = \sqrt{A^2 + B^2} = \frac{X_0}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

$$\tan\phi = \frac{-A}{B} = \frac{2\xi r}{1-r^2}$$

$$v_p = \frac{X_0}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\omega' t - \phi) \tag{2.42}$$

Since the forcing function is $p_0 \sin \omega' t$, (2.42) indicates that the effect of the forcing function is to magnify the peak deflection by a factor. This factor is the dynamic amplification factor, DAF, and is given by,

$$DAF = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \tag{2.43}$$

The comparison also shows that the response lags the forcing.

Eqn (2.43) indicates that the amplification depends solely on the dynamic properties of the SDOF oscillator – the damping ratio, and the frequency ratio. Eqn (2.42) is an atomic solution since it can be used as the basis for the solution of any periodic (though non-harmonic) forcing function, as presented in subsequent sections.

Recall that (2.32) and (2.42) are with respect to the steady-state or particular solution. When the complementary function or transient response is included, the total response (for the underdamped case) is given by,

$$v = \frac{X_0}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \sin(\omega' t - \phi) + X e^{-\xi \omega_n t} (\sin \omega_d t + \theta) \tag{2.44}$$

2.2.2.1.2 Impulse

An impulse load is one that is suddenly applied, but impulse is the product of force and time, or the area under a force-time curve. Hence if the load is to be considered at a point in time, t, we must consider the limiting case when an infinitesimal time interval around the impulse load, tends to zero.

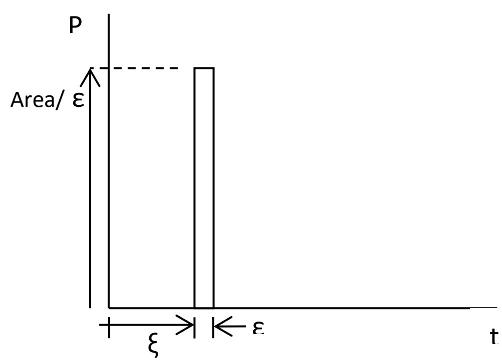

Figure 11

Clearly, as ε tends to zero, the impulse load, given by Area/ ε, tends to infinity. It cannot quite become infinity since the impulse load is finite. If the area is unity, this is termed as the unit impulse and as ε tends to zero, the corresponding force meets the definition of the Dirac Delta Function or Delta Function. The delta function has the conventional notation of $\delta(t - \xi)$ and has the properties of,

For all $t \neq \xi$, $\delta(t - \xi) = 0$

For $t = \xi$, $\delta(t - \xi)$ = a large finite number (2.45)

Also, for $0 < \xi < \infty$ $$\int_0^\infty \delta(t - \varsigma)\, dt = 1.0 \tag{2.46}$$

$$\int_{0}^{\infty} f(t)\,\delta(t-\varsigma)\,dt = f(\varsigma) \tag{2.47}$$

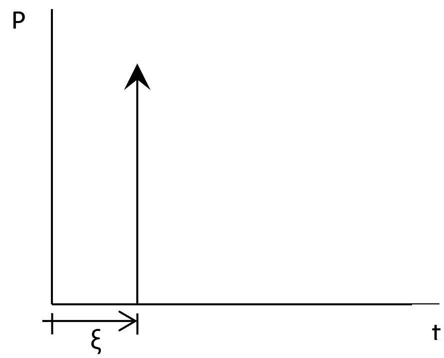

Figure 12

Because of eqn (2.45), the impulse load, represented by the delta function, is graphically depicted as an arrow at the location where it has a value (i.e. t = ξ). Note also that the notation for the delta function involves (t - ξ). This is meant to indicate that an impulse has an effect for values of t beyond t = ξ, as in a pond that continues to ripple long after a stone is thrown into it.

An impulse P_t acing on a system will cause a sudden change in its velocity without a change in its displacement during the infinitesimal time during which it is applied. From Newton's Second Law, this change in velocity equals P_t/M. As velocity is dv/dt and the impulse is applied at t = 0,

$$(dv/dt)_{t=0} = P_t/M \tag{2.48}$$

Furthermore, at t=0 when the impulse is applied, v = 0, so v(0) = 0. Together with (2.48), these are the initial conditions for the impulse vibration. We may recognize that this is a special case of the initial conditions of free vibration hence we can utilize those solutions to determine the response of an SDOF system under impulse load. Hence, for undamped vibration, from (2.24)

$$v = \frac{P_t}{M\omega_n}\sin\omega_n t = P_t\,h(t) \tag{2.49}$$

For the case of damped vibration, from (2.31)

$$v = e^{-\xi\omega_n t}\,\frac{P_t}{M\omega_d}\sin\omega_d t = P_t\,h(t) \tag{2.50}$$

In both (2.49) and (2.50), h(t) is the <u>response</u> to a unit impulse.

2.2.2.1.3 Periodic

Figure 5 above shows an example of a periodic load. In that section it was shown that such a load can be represented by the Fourier Series (FS). Now that the solution for the response of a linear SDOF system under general harmonic loading is known, the Principle of Superposition allows determination of the solution for periodic loading as the sum of the (steady-state) solutions for general harmonic loading. This is because the FS representation of the periodic loading is expressed as a sum of general harmonics.

Hence recalling eqn (2.21),

$$M\frac{d^2v}{dt^2}+C\frac{dv}{dt}+Kv=F(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\omega' t+b_n\sin n\omega' t\right) \qquad (2.51)$$

Eqn (2.42) is the solution when the harmonic force is $p_0\sin\omega' t$ and is of the same form. Likewise, if the harmonic force is $p_0\cos\omega' t$ the solution is also of the cosine form but lagging by the same phase angle as for the sine case. Therefore the steady-state solution of (2.51) is,

$$v=\frac{a_0}{2K}+\sum_{n=1}^{\infty}\frac{a_n\cos(n\omega' t-\phi_n)+b_n\sin(n\omega' t-\phi_n)}{K\sqrt{(1-n^2r^2)^2+(2\xi nr)^2}} \qquad (2.52)$$

Note that K is required in the denominator to convert to displacement.

Given a periodic load F(t), the solution is therefore obtained by first representing the load as a FS, and substituting the Fourier coefficients into (2.52).

2.2.2.1.4 Non-Periodic

Non-periodic dynamic loading, also known as arbitrary dynamic loading, is the most general type of loading. Examples are earthquake loading on a building, blast loading, or collision loading. Solutions for this type of loading are based on the solution or response for a unit impulse, h(t), which is why the latter is considered an atomic solution.

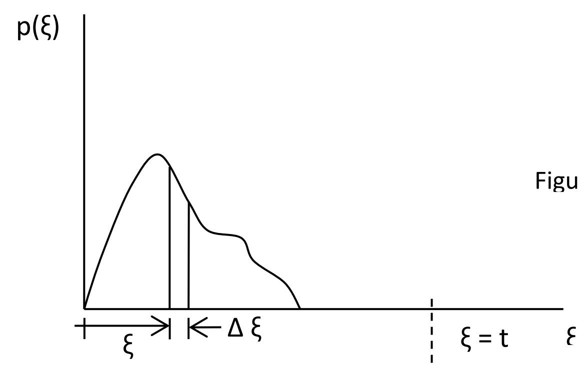

Figure 13(a)

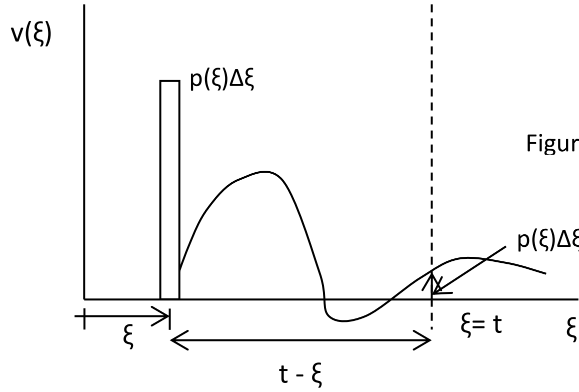

Figure 13(b)

Figure 13(a) depicts the following situation – at a point in time ξ in an arbitrary loading function, the load can be considered an impulse load. In Figure 13(b) - the curve is the response due to the impulse in (a) hence occurs after ξ has elapsed (i.e. t- ξ). Since the impulse associated with p(ξ) is p(ξ)Δξ, and the response due to a unit impulse is h(t- ξ), the response due to p(ξ)Δξ must be p(ξ)Δξ h(t- ξ).

However, this response is due to the load at a single point. If the entire arbitrary loading function is considered a set of impulses, the solution at time t for this set of impulses can be obtained by superimposing the responses for all the impulses. Hence in the limit, this superimposition or summation, becomes an integral. Therefore,

$$v(t) = \int_0^t p(\varsigma) h(t-\varsigma)\, d\varsigma \tag{2.53}$$

Eqn (2.53) is the displacement response for arbitrary loading, where h(t- ξ) is given by (2.49) and (2.50) for the case of undamped and damped systems respectively. Eqn (2.53) is called the convolution integral or the superposition integral.

Hence for the damped case, the solution for arbitrary loading under zero initial conditions is given by,

$$v(t) = \frac{1}{M\omega_d} \int_0^t p(\varsigma) e^{-\xi\omega(t-\varsigma)} \sin\omega_d (t-\varsigma)\, d\varsigma \tag{2.54}$$

For non-zero initial conditions, the complete solution is the superposition of the particular solution and complementary function. Therefore including the damped free vibration solution of eqn (2.31), the complete solution is,

$$v(t) = e^{-\xi\omega t}\left(v(0)\cos\omega_d t + \frac{\left(\frac{dv}{dt}\right)_{t=0} + \xi\omega v(0)}{\omega_d} \sin\omega_d t \right) + \frac{1}{M\omega_d} \int_0^t p(\varsigma) e^{-\xi\omega(t-\varsigma)} \sin\omega_d (t-\varsigma)\, d\varsigma \tag{2.55}$$

Response Spectrum
A response spectrum is a plot of maximum or peak response versus frequency and is therefore of great interest to the engineer for design applications. In the time domain, the convolution integral can be used to calculate response spectra. This approach to determining response spectra is however quite tedious compared to alternative approaches, except for simple non-periodic dynamic loads such as shock loading.