2.2.2.1.1 Random

If at a particular time t_1 the value of the load is known with certainty, the load is called deterministic. This is the case for all the former types of load previously considered. However, there are many circumstances when the load value is not known with certainty but, for example, the average value and the range of values may be known. In such a case, the load is considered random. In general, when a quantity is random with time as the independent variable, it is called a random or stochastic process, and statistical methods are used to describe the load and response values.

Consider Figure 15 in which case, at a point, a number of earthquake events result in a number of load-time curves, each associated with a particular event. Alternatively, there may be one earthquake but recordings are taken at the same point on a structure, but for a number of identical structures at different locations. Each load-time curve is called a time series, or a record, and is referred to as a sample of the random process. The set of samples is called the ensemble.

There are different classes of random process. At time t_1 the average value of the load for the ensemble can be determined. This is called the ensemble average at t_1. At say t_2, the average load can also be determined, and so on for each t. If for all the t, the average load is the same, the process is called a stationary process and there is one ensemble average load value.

Alternatively, the load values for a record can be averaged for the entire time of the record. For example, if a record is 60 seconds long, the sum of all load values in the record, divided by 60, is the time average for the record.

If a process is stationary, and the time average of the load is the same for each record, <u>and this equals the ensemble average</u>, the process is then called an ergodic process. For such a process, the calculations required to determine the statistical properties of the process, can be performed using only one record from the ensemble. In the following description of random quantities of structural dynamics, it is assumed that the process is stationary and ergodic.

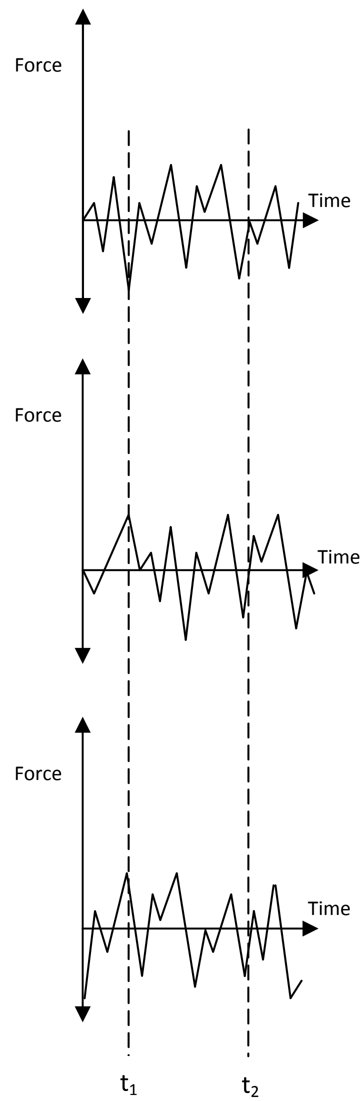

Figure 15

Expected Value and Mean Square Value

Statistical methods are used for dynamics problems involving random processes. The normal distribution is frequently utilized and this requires the mean value and standard deviation for is complete description.

The mean value of the time series sampled over a long time, also known as the expected value, is defined as,

$$E[x(t)] = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt \tag{2.66}$$

In the case of discrete variables x_i, the expected values is given by,

$$\bar{x}(t) = E[x(t)] = \lim_{T\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i \; dt \tag{2.67}$$

The mean-square value of a time series is associated with its energy and can be calculated for any time-varying load, deterministic or random. However, the standard deviation of a random process can be determined from the mean square value. The mean square values is defined as,

$$E[x^2(t)] = \overline{x^2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2 \, dt \qquad (2.68)$$

The variance is the square of the standard deviation and is defined as,

$$\sigma^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T (x - \bar{x})^2 \, dt \qquad (2.69)$$

Hence,

$$\sigma^2 = \overline{x^2} - (\bar{x})^2 \qquad (2.70)$$

The mean square value of a record is related to the Fourier coefficients of the FS of the record, when the FS is expressed in exponential (i.e. complex) form (refer to section 2.1.1.2). Considering a one-sided spectrum, let the Fourier coefficients be denoted P_n and its complex conjugate, P_n^*. Then,

$$\overline{x^2} = \sum_{n=1}^{\infty} \frac{1}{2} P_n P_n^* = \sum_{n=1}^{\infty} \frac{1}{2} |P_n|^2 = \sum_{n=1}^{\infty} \overline{P_n^2} \qquad (2.71)$$

The root mean square of a record is often a quantity of interest and is the square root of eqn (2.71).

Power Spectrum and Power Spectral Density

Recall that eqn (2.21) is the governing equation for the SDOF vibration for any deterministic load F(t). When the load is random, the design quantities of interest are random variables, and the governing equation is expressed in terms of those variables. In such a case, the load is described in terms of its power spectrum. Solution of the associated stochastic differential equation is beyond the scope of this text, but the definition of the power spectrum and the power spectral density is instructive, and a necessary foundation for further study of the topic of random vibrations. Ultimately, the aim of random vibration studies is he calculation of the probability of failure of the structural system. This is called the reliability of the design.

The term within the summation of eqn (2.71) is the definition of the power spectrum, $G(f_n)$, within the frequency interval Δf. Hence,

$$G(f_n) = \frac{1}{2} P_n P_n^* \qquad (2.72)$$

Therefore, the mean square value is the sum of the power spectrum over all frequencies.

$$\overline{x^2} = \sum_{n=1}^{\infty} G(f_n) \qquad (2.73)$$

The concept of the "power" of the time series can be generalized from the power spectrum (PS). As indicated by eqn (2.72), the power spectrum has different values for different frequencies. However, if the "power" is defined in terms of the power spectral density (PSD), the PSD can be constant for the entire frequency range, though the PS varies over the same frequency range.

This means that the PSD, as a higher generalization, can be more useful in characterizing random vibrations, and the solutions of such dynamics problems. The discrete PSD, denoted as $S(f_n)$, is defined as,

$$S(f_n) = \frac{G(f_n)}{\Delta f} = \frac{P_n P_n^*}{2\Delta f} \tag{2.74}$$

Hence the mean square value of the time series is,

$$\overline{x^2} = \sum_{n=1}^{\infty} S(f_n)\Delta f \tag{2.75}$$

Hence also,

$$S(f_n) = \lim_{\Delta f \to 0} \frac{\Delta(\overline{x})^2}{\Delta f} \tag{2.76}$$

A graphical example of a discrete PSD is shown in Figure 16.

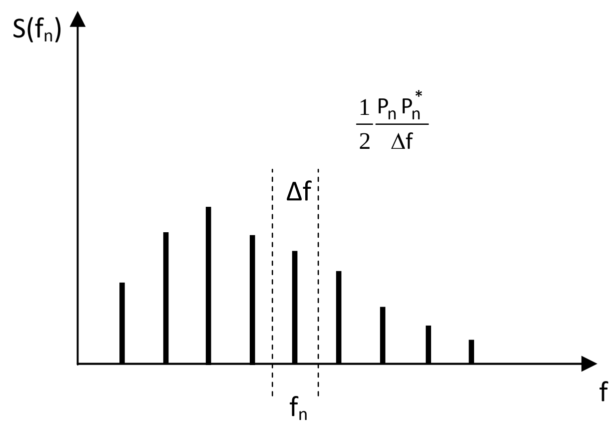

Figure 16

The PSD of a continuous (as opposed to a discrete) spectrum is defined as,

$$\lim_{\Delta f \to 0} = S(f_n) = S(f) \tag{2.77}$$

Therefore, the mean square value in terms of the continuous PSD is given by,

$$\overline{x^2} = \int_0^{\infty} S(f)\, df \tag{2.78}$$

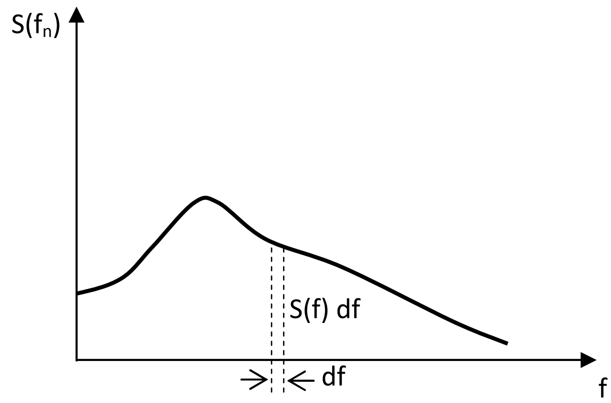

Figure 17

A graphical example of a continuous PSD is shown in Figure 17.

Vibrations are called wide-band or narrow-band. If there are many frequencies in the record, it may be termed "wide-band", such as in Figure 17. This means that the PSD curve covers a wide range of frequencies. On the other hand, a system vibrating near resonance would display a narrow-band PSD, as shown in Figure 18. When the PSD of a record is wide-band and constant, the vibration is referred to a white-noise.

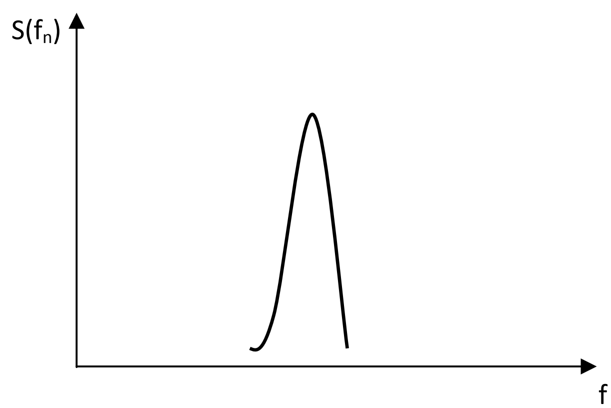

Figure 18

2.2.2.1.2 The Fast Fourier Transform

In section 2.1.2, the Fourier Transform (FT) was presented. Given the time series, the FT provides the Fourier coefficients of the exponential form of the FS representation of the time series. For a frequency n, this coefficient is the magnitude of the time series at that frequency. If the associated integration can be performed in closed-form (i.e. the result of the integration is a function), the FT is described as the direct FT.

However, many time series of practical interest are not integrable in closed-form and in this case numerical methods must be used. It is important to note that in general, numerical methods also provide solutions for problems that can be solved in closed-form.

The application of numerical methods to the FT results in the procedure or algorithm called the discrete FT or DFT. In this approach, the input consists of N data points as pairs of (time, signal) values [i.e. (time, load) if the time series is with respect to load]. The algorithm then computes a set of N complex values X_k, that represent the frequency domain information, or sinusoidal decomposition of the time series. Each pair of the input data is called a sample. This is because it is typically obtained from digital instruments that record data at a certain sampling rate.

Like all numerical methods, the DFT is applied via a computer program. However in general, the DFT can be time-consuming and this lead to the development of a form of DFT called the Fast Fourier Transform (FFT). The FFT is based on the observation that if the number of data pairs is a power of 2, that is $N = 2^M$, then the DFT is performed much faster. However, certain conditions must be met in order to avoid errors arising from the FFT procedure.

Common Errors of the FFT

1. Aliasing

   "Aliasing" refers to the phenomenon of an item in the FFT output not appearing as it is, but rather appearing elsewhere and possibly combined with a legitimate output data point. Aliasing occurs due to the fact that though the physical signal is analogue, it is measured digitally by sampling at a certain frequency, and this introduces periodicity to the sampled data. This means that the values can only be unique if in a certain frequency range. For frequencies beyond that range, the data is folded-back into the range. The frequency at which this fold-back occurs is called the Nyquist frequency. This means that the FFT can only correctly depict the frequency content of a time series if the highest frequency in the time series is less than the Nyquist frequency. The Nyquist Frequency is equal to half the sampling frequency. Since the time series or signal is sampled every T seconds, there are 1/T samples per second, hence the sample frequency is 1/T.

   The FFT output values are separated by a frequency interval of 1/(NT) Hz hence the k^{th} value corresponds to a frequency of k/(NT) Hz. Therefore, if there are 64 (i.e. 2^6) samples that were sampled at 800 Hz, then T equals 1/800 seconds, and N equals 64, so the FFT outputs values at frequencies, in ascending order, of 0 Hz, 1/((1/800)x64) = 12.5 Hz, 2/((1/800)x64) = 25.0 Hz, 3/((1/800)x64) = 37.5 Hz, etc. The Nyquist Frequency is 1/(2T) = 1/(2x(1/800)) = 400 Hz and will correspond to the 32^{nd} value therefore data for points beyond the 31^{st} should be ignored.

   To avoid aliasing the following measures can be taken:

   a. Use a sample frequency greater than twice the highest frequency in the time series.
   b. Pre-filter the time series with an anti-aliasing filter to ensure there is no high frequency data in the time series. Of course, this data must be unimportant.
   c. A combination of a and b, but first filter and then over-sample, or the aliasing effect will not be removed.

2. Leakage

   Closely-spaced magnitude values in the FFT output may indicate the occurrence of leakage and these values can be significantly far from the true value. Leakage occurs when a frequency of the time series that has a value, does not fall exactly on one of the points in the FFT output. If this is observed, the number of points in the sample can be adjusted.

3. Overshoot

   This is also known as the Gibbs phenomenon. It refers to the fact that if the time series has a discontinuity, such as the vertical lines of a square wave, then at the corners of the discontinuity the inverse FFT will give higher values than reality. This phenomenon is not removed by increasing the sample rate, but such increase has the effect of reducing the length of the zone over which it happens. Hence this effect must be considered for problems involving impact loads as they are discontinuous.