2.2.2 Non-Linear

Reproducing eqn (2.21),

$$M\frac{d^2v(t)}{dt^2}+C\frac{dv(t)}{dt}+Kv(t)=F(t)$$

In this equation, the C and K are constant but there are several cases when this is not so. For example, in the third term on the LHS - the spring force term, the stiffness K is a constant because the load-deformation relationship for the structure is linear. However, if K depends on the deformation of the structure v, the term Kv becomes K(v)v which is clearly nonlinear, and this makes the entire equation nonlinear.

A common example of such nonlinearity is inelasticity in which case the load-deformation relationship does not obey Hooke's Law. For instance, if a structure is loaded such that yielding occurs within the structure, plastic deformation takes place. One of the central concerns of engineers working in earthquake prone regions is the dynamics of structures that have entered the plastic range since this is the main approach for the earthquake resistant design of structures. Though there are other types of nonlinear equations of motion in structural dynamics, the type due to the earthquake loading of structures in the plastic range of response is the type considered in this text.

The solution of a nonlinear dynamics equation cannot be obtained in closed-form, such as for the several cases considered earlier. Recall that for those cases, the solution method is the symbolic integration of the equation of motion, which is a linear differential equation. As in many other areas of engineering mechanics, when a closed-form approach cannot be used, a numerical method can provide a solution. Furthermore, even if the equation is linear, solution in closed-form generally cannot be obtained if the dynamic loading is arbitrary. Therefore, numerical integration is required for the two situations of: (1) linear structures under arbitrary loading, or (2) plastic response. Under earthquake loading, both situations simultaneously occur for many structures of interest.

In this section the numerical solution of the equation of motion via a popular method is presented and is applied to the case of arbitrary loading of a linear structure. The same method can be applied to the case of the nonlinear equation.

2.2.2.1 Numerical Integration of the Equations of Motion

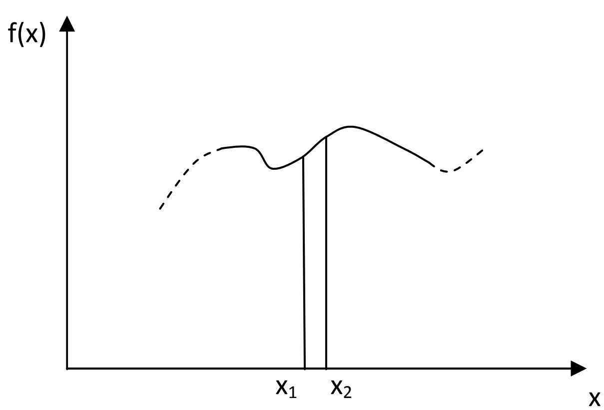

Figure 19

Figure 19 shows a portion of a general function v=f(x). We know a relationship comprised of derivatives of v but desire to know v for all x. If we know the value of $f(x_1)$, and the slope of f(x) between x_1 and x_2, we can approximately calculate f(x) at x_2. Note that the slope is the first derivative of f(x). A better approximation is obtained if higher derivatives are used since in the most general case they may exist and if so, contribute to f(x).

The Taylor series gives the most general representation of f(x) in terms of its derivatives, but the series is infinite. In numerical integration, higher derivatives beyond a certain selected term in the Taylor series are typically ignored (truncated) and in many cases, the others are replaced by averaged first derivatives or slopes. Truncated series are the basis of all numerical methods therefore all numerical methods give approximate results. However, the methods typically give reasonably accurate results and most importantly, are readily applicable to simple computation, especially via computers. In these methods the differential equation is effectively replaced by a set of algebraic equations and the computation starts from x=0 and proceeds step-by-step to the next x at end of the interval "h" (e.g. $x_2 - x_1$ in Figure 19), but using the results of the previous step. The values at x = 0 must be known. The calculation is therefore iterative in nature and called step-by-step direct integration.

There are several classes of numerical integration methods. These were developed so early compared to the history of computers that a very important criterion in the selection of a method was the time required by the individual or computer to run the solution. In the earlier days of the evolution of computers the final decision was frequently based on this criterion only, since differences in expensive computing time was typically higher than differences in accuracy. The high speed and relatively low cost of modern computers has changed this scenario so in civil engineering relative accuracy is probably the main criterion with the result that the Runge-Kutta 4^{th} Order method (RK4) is a popular choice, named after the German mathematicians who developed it.

The order of a method is a measure of its accuracy. The error associated with a method is proportional to its order "m" which is the power to which the interval "h" is raised (i.e $O(h^m)$) . "h" is less than unity so the higher the order, the smaller the error. The RK4 method is classified as a self-starting, unconditionally stable, single-step method. There are 2^{nd} and 3^{rd} order RK methods but the classical RK method is the 4^{th} order, due to its generally better balance of computing time versus accuracy. There is also a 5^{th} order RK method that is more accurate but more time consuming.

In the RK4 method, the higher derivatives of the Taylor series are replaced by weighted averages of the slope in the interval. The RK methods integrate a first order differential equation. However one of its advantages is that by suitable substitution, second and higher order differential equations, or sets of such equations can be replaced by a set of simultaneous first order differential equations then the RK method applied.

Mathematics texts can be consulted for the derivation of the RK4 method but the concern of this text is its algorithm which is based on the following fundamental equations.

Let $\ddot{v} = \frac{d^2v}{dt^2}$ and $\dot{v} = \frac{dv}{dt}$. Note that in terms of notation, the independent variable "x" has been replaced by "t" for application to dynamics.

$$\ddot{v} = f(t, v, \dot{v})$$

Let $\dot{v} = y$,

hence $\dot{y} = \ddot{v}$

$$v(t_i + h) = v(t_i) + h\left[\dot{v}(t_i) + (k_1 + k_2 + k_3)/6\ \right] \quad (2.79)$$

$$\dot{v}(t_i + h) = \dot{v}(t_i) + (k_1 + 2k_2 + 2k_3 + k_4)/6 \quad (2.80)$$

$$k_1 = h\ f(t_i, v_i, \dot{v}_i) \quad (2.81)$$

$$k_2 = h\ f(t_i + \frac{h}{2}, v_i + \frac{h}{2}\dot{v}_i + \frac{h}{8}k_1, \dot{v}_i + \frac{k_1}{2}) \quad (2.82)$$

$$k_3 = h\ f(t_i + \frac{h}{2}, v_i + \frac{h}{2}\dot{v}_i + \frac{h}{8}k_2, \dot{v}_i + \frac{k_2}{2}) \quad (2.83)$$

$$k_4 = h\ f(t_i + h, v_i + h\dot{v}_i + \frac{h}{2}k_3, \dot{v}_i + k_3) \quad (2.84)$$

It is quite simple to write a computer program for eqns (2.79) to (2.84) since they depend on evaluations of just one function at different points in the interval. A function can be defined whose arguments enable evaluation at any point and the main program will merely define those points and call the function as required.

The algorithm for the RK4 method can also be implemented in a form amenable to hand or spreadsheet calculation. In the approach by Thomson, a 4x4 grid is used to implement the calculations given by (2.81) to (2.84) for each time value. These intermediate values are then substituted in (2.79) and (2.80), and the equation for $\ddot{v}$, to obtain the solution at t_i. "t" is then incremented and the process repeated until the end. The 4x4 grid of intermediate calculations is given by –

t	v	y= $\dot{v}$	f
T1 = t_i	X1 = x_i	Y1 = y_i	F1 = f(T_1, X_1, Y_1)
T2 = t_i + h/2	X2 = x_i + Y1 x h/2	Y2 = y_i + F_1 h/2	F2 = f(T_2, X_2, Y_2)
T3 = t_i + h/2	X3 = x_i + Y2 x h/2	Y3 = y_i + F_2 h/2	F3 = f(T_3, X_3, Y_3)
T4 = t_i + h	X4 = x_i + Y3 x h	Y4 = y_i + F_3 h	F4 = f(T_4, X_4, Y_4)

Then given (2.79) and (2.80) and some modification,

$v_{i+1} = v_i + h/6(Y1 + 2Y2 + 2Y3 + Y4)$ (2.85)
$\dot{v}_{i+1} = \dot{v}_i + h/6(F1 + 2F2 + 2F3 + F4)$ (2.86)

The following is a sample problem.

For the first three iterations only, solve the equation, $\ddot{v} = 0.3p(t) - 200v$ under initial conditions v = dv/dt = 0, and for p(t) = 300 for all t. Use h = 0.02 sec.

t	v	dv/dt	f
0.00	**0.00**	**0.00**	**90.00**
0.01	0.00	0.90	90.00
0.01	0.01	0.90	88.20
0.02	0.02	1.76	86.40
0.02	**0.01788**	**1.776**	**86.424**
0.03	0.04	2.64	82.87
0.03	0.04	2.60	81.14
0.04	0.07	3.40	76.01
0.04	**0.070096**	**3.4108672**	**75.9808064**

The data in boldface are the required answers – the first row of the grid of the current iteration given by eqn (2.85) and (2.86), except for the first row which are the values at t = 0. They are the desired values at t_i. The next three rows are the rest of the grid.

Response Spectrum
Due to the versatility of the computer, the numerical integration can be used as the main part of a program for calculating the response spectrum. Though the integration is in the time domain, the M and/or K values can be changed thereby altering the period or frequency, and for each point the loading is applied as a function of “t”, and from the results the peak response is noted and stored in a file. The process is then repeated for another period until a range of periods is covered.

2.2.2.2 Inelastic Load-Deformation (Time Domain)

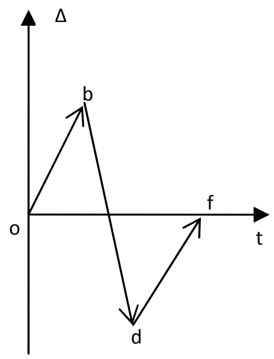

Figure 20(a)

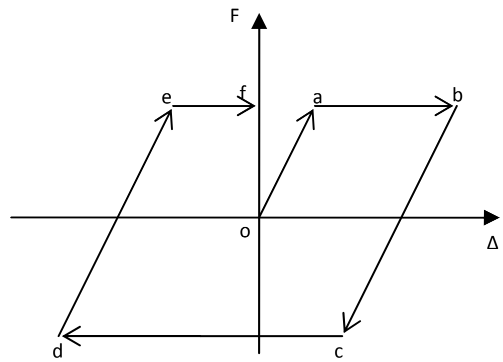

Figure 20(b)

Consider an experiment performed as shown in Figure 20(a) above. A lateral displacement is applied to a structure from zero (at “o”) well beyond its yield point, and then the displacement is reversed (at “b”) until the same displacement is reached but in the other direction (at “d”). Then the displacement is reversed again until the structure is returned to its original position (at “f”) thus making one complete cycle. When the displacement is slowly applied, such a test is called a quasi-static or pseudo-static test, and Figure 20(a) is called the loading protocol. If a test is done without reversing the applied displacement, for example from o-b only in Figure 20(a), the test is called a monotonic test.

If the resistance is measured simultaneously as the displacement is applied, then a plot of the resistance versus applied displacement can be traced out as the test progresses, as indicated by o-a-b-c-d-e-f in Figure 20(b). Such a shape is called a hysteresis loop. The enclosed area of the loop is the energy absorbed in the structure at the locations undergoing internal plastic deformation. The loop occurs due to the permanent set or irreversible deformation that accompanies plastic flow.

The loop shape indicated in Figure 20(b) is the simplest type of cyclic load-deformation shape possible, and is called elastic perfectly-plastic (EPP). A higher type of deformation, though quite simple, is when the a-b and c-d are not horizontal but inclined to the deformation axis, thus indicating strain-hardening plasticity. In such a case, the shape is called bi-linear (BL), and therefore the EPP is a special case of bilinear hysteresis.

The BL and EPP shapes are idealizations of the real nonlinear cyclic or dynamic behavior of structural systems but they enable more simplified equations for design use. In real nonlinear cyclic or dynamic behavior, the strength of the system (i.e. the load values along a-b and c-d) changes as the applied displacement continues, as well as the slopes of b-c and d-e. These phenomena are called strength degradation and stiffness degradation, respectively. Other important phenomena can occur as well, and interestingly, the degree of degradation at any time depends on the entire history of degradation before that time. Each type of structural system has its own characteristic loop shape and degradation types, and degradation rates.

As discussed earlier, the significance of the inelastic behavior is that it changes the differential equation of motion as the spring force term is now dependant on the displacement "v" in a non-simple manner.
Representation of the Nonlinear Spring Term

The nonlinear equation of motion is given by,

$$M\frac{d^2v(t)}{dt^2}+C\frac{dv(t)}{dt}+K(v)v(t)=F(t) \qquad (2.87)$$

Clearly, solving eqn (2.87) can only be undertaking via a computer program in which it is necessary to define K(v)v(t). There are two ways of doing this – (a) the rule-based approach, and (b) the mathematical approach.

The Rule-Based Approach
In the rule-based approach, a function is defined in the computer program that determines what value the load should be at the current displacement depending on where it was at the last displacement. For example, in Figure 20(b), a rule could be "if the displacement is higher than the displacement at yield, the force is the strength of the system". A line is drawn connecting this point to the previous point. Therefore, rule-based approaches to describing K(v)v(t) result in load-deformation curves that are comprised of straight-line segments. Such curves are called "piecewise linear". A well-known rule-based model for reinforced concrete systems has 14 rules and is called the Takeda model.

The Mathematical Approach
In the mathematical approach K(v)v(t) is defined using formulae. From earlier studies, one such model for steel is the Ramberg-Osgood model. In more recent times, a particular differential equation model called the Bouc-Wen model, is used extensively. In these mathematical models, there are a number of parameters that can be determined from test results. Another noteworthy differential equation model with fewer parameters is the Da model which is presented herein. Some of the advantages of the mathematical approach are that the resulting load-deformation curves are smoothly-varying which is more realistic in comparison with the piecewise linear plots. As they are more compact, dynamics programs that use them are more efficient and easy to maintain, with this efficiency being much larger when applied to the study of random nonlinear dynamics, especially in earthquake engineering.

The Da Model describes the cyclic load-deformation relationship for the SDOF system as a differential equation. It is given by eqns (2.89) and (2.90), and drives the governing equation (2.88).

$$\ddot{X}+\frac{1}{m}(C\dot{X}+ZF_p/U_p)=-a_g/U_p \qquad (2.88)$$

where,

$$\frac{dZ}{dX}=\left(\frac{K\,U_p}{F_p}\right)\left(1-D(X,Z)\right) \qquad (2.89)$$

and,

$$D(X,Z)=\frac{p_1}{p_2+p_3}\left[p_2\,\mathrm{sgn}(dX)\frac{Z}{|Z|}\;|\sinh(p_4F_pZ)|+p_3|\sinh(p_4F_pZ)|\right] \qquad (2.90)$$

Z and X are the normalized force (F/F_p) and corresponding lateral displacement (U/U_p). F_p and U_p are with respect to the peak restoring force point of the hysteresis envelope or backbone curve so F_p is the capacity or strength of the structural system, and U_p is the corresponding displacement at that point. K is the initial stiffness. p_1 to p_4 are constants such that $p_2 + p_3 = 1$ and $p_4 = 1$.

$$p_1 = \frac{3}{\sqrt{(3e^{2F_p} - 2)} - 1} \tag{2.91}$$

Eqn (2.88) can be solved by the Runge-Kutta 4th order numerical integration procedure (RK4). Let,

$$y_1 = X \tag{2.92}$$

$$y_2 = \dot{X} \tag{2.93}$$

$$y_3 = Z \tag{2.94}$$

$$y_4 = \varepsilon \tag{2.95}$$

Hence,

$$\dot{y}_1 = \dot{X} \tag{2.96}$$

$$\dot{y}_2 = -\frac{1}{m}(Cy_2 + y_3 F_p / U_p) - a_g / U_p \tag{2.97}$$

$$\dot{y}_3 = y_2 D \tag{2.98}$$

$$\dot{y}_4 = y_2 y_3 \tag{2.99}$$