# 3 MDOF Lumped Parameter Dynamics

Thus far, the approach to structural dynamics has been based on idealizing the entire structure as a SDOF system. This implies that the structure, in three dimensions, is uniform with respect to the important dynamic parameters of its mass and stiffness (and for nonlinear structures its strength). However, in many practical situations this is not the case.

The aim of analysis in consideration of dynamics effects is to determine the effect of amplification due to the mobilization of inertia due to the time-varying nature of the applied loading. This amplification is related to the distribution of stiffness and mass throughout the structure.

Therefore, it is the responsibility of the analyst to ensure that the model of the structure is representative of the actual given structure. This is the essential initial stage of any dynamics problem and is termed "modeling" (for dynamic analysis). From this, the governing equations can be formulated, and a solution of those equations sought.

In this section, the simplest type of MDOF modeling is considered, yet the approach applies in general to more complex physical cases. When the load-deformation relationship is linear, a particularly useful method is "modal analysis" hence this technique is presented herein. In modal analysis, the aim is to uncouple the essentially coupled simultaneous differential governing equations of the structure when modeled as an MDOF structure.

Implicit in modal analysis, is the determination of the natural frequencies and mode shapes of the structure. This is the solution to the undamped free vibration solution to the MDOF governing equations. These topics are therefore the subject of this chapter.

## 3.1 Modeling

As noted above, the analyst attempts to ensure that the model of the structure is representative of the actual given structure. This can be done in a number of ways and in the ideal situation, only the least complexity required for reasonably accurate output is considered.

The degrees-of-freedom are those associated with the motion of the mass of the structure hence the analyst's first consideration is the actual mass distribution of the structure. The mass can be idealized as lumped or distributed (also called continuous). In previous chapters, the lumped SDOF idealization was used. In this chapter the lumped MDOF idealization is made, and in Chapter 4, the structure is considered as a distributed mass system.

In former chapters, the mass was associated with only one DOF. However, this is not a necessity. For example, a rigid floor of a building, though it can be idealized as one mass, can also be idealized to move in translation along the x and y plan directions, as well as in rotation about the vertical axis (i.e. torsionally). This is called the kinematic idealization. Therefore in such an idealization, the mass has three DOFs. Once the mass and associated DOFs are defined, the analyst applies the physical laws (e.g. Newton's forces, Hamilton's energy, etc), to the model, in order to formulate the equations of motion for solution.

## 3.2 Solution of Linear Problems

When a structure is idealized as an MDOF (lumped mass) system, the result is a set of simultaneous differential governing equations on motion for the system. These equations can then be solved in the typical manner for coupled differential equations. However, if the load-deformation relations for the system are linear, there is a simpler approach.

In this approach, the analyst, in effect, reuses solutions for the SDOF case. This is done by uncoupling the simultaneous differential equations, to form a set of independent SDOF equations of known solution. This approach is called modal analysis or modal superposition. If the SDOF solutions as functions of time are used, the

approach is called time history or response history analysis (RHA). As a subset of this, if only the peak SDOF response solution are used (i.e. the response spectrum solutions), the analysis technique is called response history analysis (RHA).

### 3.2.2 Principles of Modal Analysis

It is convenient to express the MDOF governing equations of motion in matrix form. When this is done, the property matrices (i.e. [M], [C], and [K]) are typically fully populated since they represent simultaneous equations. Therefore to solve them, the laborious matrix inversion must be undertaken. Independent equations do not require inversion since they are in effect as simple as 3x = 6. Hence, it is desirable to convert the governing equations into a form in which they are independent. This is called "uncoupling" the equations.

When equations are independent, if they are represented in matrix form, it is observed that the coefficient matrix only has non-zero values along the diagonal of the matrix. Therefore, to uncouple the equations a means for converting a general matrix to a diagonal matrix is required.

In Linear Algebra, this is achieved using the Diagonalization Theorem. This states that a general matrix will be converted to a diagonal matrix if it is pre-multiplied and post-multiplied by $[\phi]^T$ and $[\phi]$, respectively, if $[\phi]$ is an orthogonal matrix. An orthogonal matrix is one whose inverse equals its transpose.

Therefore, in modal analysis use is made of the Diagonalization Theorem hence the orthogonal $[\phi]$ matrix. Also from Linear Algebra, any matrix that is the solution of the eigenproblem, is an orthogonal matrix. In structural dynamics, the undamped free vibration problem of the MDOF system is an eigenproblem hence its solution is the required $[\phi]$ matrix. The $[\phi]$ matrix is therefore the displacement solution for the MDOF system when it is undamped and vibrates freely.

If the MDOF governing equations in matrix form (which is square) is of order "n", then its free vibration has "n" solutions. Each solution is called a "mode shape" and each mode shape vibrates at a particular frequency. The $[\phi]$ matrix is an n x n matrix and each column of the matrix contains the displacement values of a particular mode shape. The $[\phi]$ matrix is therefore called the mode shape matrix.

The principles of Modal Analysis can be summarized as the following steps:-

1. Determine the modes of vibration of the system, $[\phi]$.
2. Represent the displacement of the masses as the sum of the modal displacements times a Y vector, called the *general co-ordinate vector,* or the *normal coordinates*. (i.e. $[v] = [\phi][Y]$).
3. Make the system of equations independent of each other so that the system becomes a set of independent SDOFs.
4. Use the solutions to SDOF problems to determine the response for each mode.
5. Superimpose these results in accordance with $[v] = [\phi][Y]$.

Consider steps 2 to 4 above in greater detail as follows.

After performing the mass modeling, and putting this and the structure's stiffness in matrix form, we get

$$[M][\ddot{v}] + [C][\dot{v}] + [K][v] = [F(t)] \qquad (3.1)$$

where [C] is the classical damping matrix. The [M], [C] and [K] matrices are n x n matrices, and the [F(t)] matrix is a n x 1 matrix, or vector, that contains the functions for the applied dynamic loads, one at each mass.

Expressing the displacements "v" in terms of the normal coordinates "$\phi$Y",

$$[v] = [\phi]\ [Y] \tag{3.2}$$

Hence eqn (3.1) becomes,

$$[M]\ [\phi]\ [\ddot{Y}] + [C]\ [\phi]\ [\dot{Y}]\ ] + [K]\ [\phi]\ [Y] = [F(t)] \tag{3.3}$$

After diagonalization (i.e. pre-multiplying (3.3) by $[\phi]^T$), we get

$$[M]_{dia}\,[\ddot{Y}] + [C]_{dia}\,[\dot{Y}]\ ] + [K]_{dia}\,[Y] = [\phi]^T\,[F(t)] = [P(t)] \tag{3.4}$$

Eqn (3.4) is therefore a set of uncoupled or independent equations, so dividing each by its associated mass,

$$\ddot{Y}_n + 2\xi_n\omega_n + \omega^2_n Y_n = P_n(t)/M_n \tag{3.5}$$

where "n" refers to the $n^{th}$ mode shape.

Since eqn (3.5) is the equation of motion for an SDOF oscillator, we can now use the SDOF solutions and by the superposition of (3.2), obtain the solution for the MDOF equation of motion given by (3.1).

Hence,

$$[v(t)] = \sum_{n=1}^{N} [\phi]_n\,[Y(t)]_n \tag{3.6}$$

where [v(t)] is the n x 1 column vector of displacements, $[\phi]_n$ is the $n^{th}$ row (i.e. 1 x n) of the [ϕ] matrix, $[Y]_n$ is the n x 1 column vector of SDOF solutions to eqn (3.5), and N is the total number of mode shapes. The solutions for the velocities and accelerations are similar to eqn (3.6).

For example, if there are 3 mode shapes, then expanding eqn (3.6),

$v(t)_1 = \phi_{11}\,Y(t)_1 + \phi_{12}\,Y(t)_2 + \phi_{13}\,Y(t)_3$
$v(t)_2 = \phi_{21}\,Y(t)_1 + \phi_{22}\,Y(t)_2 + \phi_{23}\,Y(t)_3$
$v(t)_3 = \phi_{31}\,Y(t)_1 + \phi_{32}\,Y(t)_2 + \phi_{33}\,Y(t)_3$

Note that tough (3.6) indicates that the responses for all modes are required for the exact solution, usually only a few are required for a reasonably accurate solution.

A subset of eqns (3.1) to (3.6) is the calculation of the peak responses, which is of interest to design engineers. In this case, the SDOF response spectra solutions are used in (3.5) and (3.6), hence the method is called Response Spectrum Analysis. Application to the case of the earthquake response of a building is presented in a subsequent chapter.

3.2.2.1 The Eigenproblem

As noted in the last section, the first step in Modal Analysis is the determination of the mode shape matrix, [ϕ], which is essential to the method because it enables the uncoupling of the simultaneous equations of motion.

This is possible via the Diagonalization Theorem because the [ϕ] matrix is orthogonal by virtue of its being the solution to the eigenproblem. The eigenproblem is well-known throughout applied mathematics and in structural dynamics is the solution of the equations of motion for the undamped free vibration of an MDOF system.

$$[M][\ddot{v}] + [K][v] = [0] \tag{3.7}$$

This equation is solved by $[v] = [v'] \sin \omega t$ (3.8)

Substitution into (3.7) gives,

$$\left([K] - \omega^2 [M]\right)[v'] = [0] \tag{3.9}$$

Dividing by [M] and letting $\lambda = \omega^2$,

$$\left([D] - \lambda [I]\right)[v'] = [0] \tag{3.10}$$

$$[I] = [M]^{-1}[M] \tag{3.11}$$

$$[D] = [M]^{-1}[K] \tag{3.12}$$

[I] is a unit matrix, and [D] is called the system matrix or dynamic matrix.

Eqn (3.10) implies that the determinant of the coefficient matrix on the LHS is zero.

$$\left| [D] - \lambda [I] \right| = 0 \tag{3.13}$$

When eqn (3.13) is expanded it is called the characteristic equation or the frequency equation and is given by

$$f(\lambda) = 0 \tag{3.14}$$

Eqn (3.14) is an $n^{th}$ order polynomial which therefore has n roots. Each λ root is called an eigenvalue. Eqn (3.10) indicates that each λ corresponds to a particular [v'], called the eigenvector.

Clearly, eqn (3.10) indicates that the eigenvectors are not unique. To obtain unique values, the calculated eigenvectors are scaled.

*The Standard Form of the Eigenproblem*

The eigenproblem as represented by eqn (3.10) is not the standard form used by physical scientists in general, and for which many computer program subroutines or libraries exist. Eqn (3.10) is particular to structural dynamics and being non-standard, cannot be used in those programs. This is because the dynamics matrix [D] is not symmetric.

To use such programs, the following adjustment is typically recommended. Let,

$$[Q] = [M]^{1/2} \tag{3.15}$$

$[M] = [Q]^T[Q]$ (3.16)

$[A] = Q^{-T}KQ^{-1}$ (3.17)

Then instead of expressing the eigenproblem as eqn (3.10), the standard form can be used as,

$$([A]-\lambda[I])[v'] = [0] \quad (3.18)$$

3.2.2.1.1 Approximate Determination of Natural Frequencies

Apart from its use in Modal Analysis, solving the free vibration problem is also of particular importance in civil engineering as it results in formulae for the natural frequencies or periods of MDOF systems.

The design engineer needs to know the magnification factor due to the mobilization of the inertia forces and, as seen in earlier sections, this depends on the frequency ratio - the ratio between the forcing frequency and the system's natural frequency.

Solving the eigenproblem is required for many problems of interest in civil engineering. However in many cases, it is only the first few modes that contribute significantly to the total response, especially the first mode, also called the fundamental mode. It is therefore desirable to be able to quickly determine the fundamental natural frequency. The following methods, the Rayleigh Method and the Dunkerley Method, enable approximate determination of the fundamental frequency. The former gives an upper bound estimate, and the latter, a lower bound estimate. These methods can be applied to distributed parameter systems, in addition to MDOF systems as presented below.

Rayleigh Method

The Rayleigh method is based on equating the maximum potential or strain energy of the dynamic system, with its maximum kinetic energy. The method is approximate because it depends on an initial estimate of the deflection profile of the structure, but since the calculated frequency is guaranteed to always be greater than the true value, this usually has conservative consequences, from a design standpoint. For this reason, the Rayleigh method is usually the preferred method of approximate determination of the fundamental frequency. The calculated frequency can only equal the true value if the estimated displacement profile is the true mode shape. The Rayleigh frequency for an MDOF system is given by,

$$\omega_1^2 = \frac{[B]^T[K][B]}{[B]^T[M][B]} \quad (3.19)$$

[B] is any n x 1 vector with displacement units. Eqn (3.19) can take different forms since there is flexibility in the choice of [B]. A common approach is to let [B] be the static displacements due to the weights of the masses taken as point loads. Hence (3.19) becomes,

$$\omega_1^2 = \frac{g\sum M_i v_i}{\sum M_i v_i^2} \quad (3.20)$$

where $y_i$ is the displacement at $M_i$.

The standard deflection formulae for beams under point loads are then used with the principle of superposition to determine the displacements. To refine the displacement profile to better represent the displacement under

dynamic load, the applied loads can be adjusted in proportion to the displacements of the static profile. This is reasonable because the inertia loads are proportional to the displacements. The displacements are then re-calculated with the revised loads, and substituted in (3.20).

Dunkerley Method

The Dunkerley method gives a reasonably accurate estimate of the fundamental frequency, if it is much lower than the higher frequencies of the MDOF system. This estimate is also a lower bound value compared with the true value. The method is based on the flexibility matrix and the estimated fundamental frequency can be determined from either of two formulae given by,

$$\frac{1}{\omega_1^2} = \sum_{i=1}^{n} a_{ii}\, m_i \qquad (3.22)$$

$$\frac{1}{\omega_1^2} = \sum_{i=1}^{n} \frac{1}{\omega_{ii}^{\,2}} \qquad (3.23)$$

In eqn (3.22), [a] is the inverse of the stiffness matrix. That is, $[a] = [K]^{-1}$

In eqn (3.23), $\omega_{ii}$ is the natural frequency of an equivalent SDOF system with only mass $m_i$ considered at its location in the original MDOF model. Therefore, for each mass, (3.22) is used to determine the corresponding $\omega_{ii}$ and these substituted into (3.23). The simple deflection formulae for beams can be used to determine the flexibility influence coefficients, $a_{ii}$, when (3.22) is being used as part of using (3.23). This is because of the definition of the flexibility influence coefficient, $a_{ii}$, as the deflection at i due to a unit load at i.

3.2.2.1.2 Numerical Determination of Natural Frequencies

Many structural dynamics problems of interest to civil engineers require MDOF models in which several modes, and hence natural frequencies, are significant. Solving the eigenproblem by the direct formulation and solution of the frequency equation in closed-form is feasible for $N \leq 4$. Beyond this it is necessary to use computer methods which means numerical approaches to the solution of the eigenproblem.

Computer software for solving the eigenproblem are called eigensolvers and are usually available in subroutine libraries, or functions in very high-level languages such as MATLAB, or advanced programmable scientific calculators. There are three main classes of algorithms in the solution of the eigenproblem via numerical methods. These are – vector iteration methods, matrix transformation methods, and polynomial root-finding methods.

Studies have indicated that there are conditions under which each of these classes of methods is the most appropriate approach. In this presentation however, the scope is limited to the "inverse iteration with shift" method because it readily enables the student to develop a feel for the process by hand calculation. The theoretical background of the method is left as an exercise of the student.

Inverse Iteration with Shift

$$[D] = [K']^{-1}[M] \qquad (3.24)$$

$$[K'] = [K] - \mu[M] \qquad (3.25)$$

Hence,

$$([K'] - \lambda'_i[M])\phi_i = [0] \qquad (3.26)$$

where,

$$\lambda'_i = \lambda_i - \mu \qquad (3.27)$$

$\mu$ is the shift that the name of the method refers to. The calculation procedure is as follows:

1. Select $\mu$.
2. Calculate [D].
3. Let s = 0.
4. Assume $[U]_s = [1\ 1\ 1\ 1\ \ldots 1]^T$.
   Begin iteration:
5. Calculate $[v]_{s+1} = [D]\ [U]_s$
6. In $[U]_{s+1} = \lambda'_{s+1}\ [v]_{s+1}$, determine $\lambda'_{s+1}$ by scaling so that the maximum value in $[U]_{s+1}$ is 1.0
7. Calculate $[U]_{s+1} = \lambda'_{s+1}\ [v]_{s+1}$
8. If no convergence, let s = s +1 and go to 5.

The eigenpair is ($\lambda_s$, $[U]_s$). To get another eigenpair, increment $\mu$ and re-do steps 2 to 8. It is possible that the selected $\mu$ will result in convergence to a formerly calculated eigenpair. Adjust $\mu$ until unique values are obtained. Note that because of the scaling to 1.0 in step 6, the eigenvectors [U] are not mass ortho-normal. However, the procedure can be modified to suit.