5.0 STRESS, STRAIN IN TIES AND STRUTS CROSS-SECTIONS

a. Direct Stress and Axial Strain:

In Chapter 1 we learned that ties are structural members whose ends are moving apart from each other, and struts are structural members whose ends are moving towards each other. In each case this behaviour is due to forces applied externally at the ends of the member.

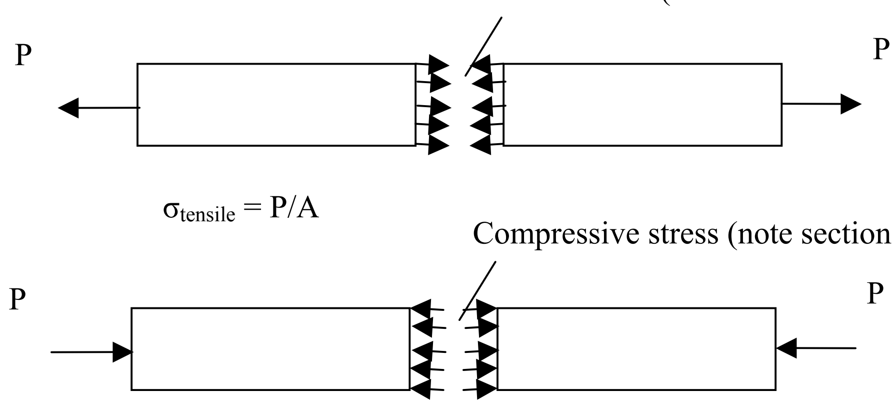
Within the member at any section internal stresses are developed which are **tensile** in the case of ties but **compressive** in the case of struts. These tensile and compressive stresses are distinguished from each other only by their direction of action, and both are called **direct** or **normal** stresses in contrast to shear stresses, which were introduced in Chapter 4. Direct stresses act on a section at right angles to the section and through its centroid.

The normal stress is simply the force divided by the area of the section. Hence,

direct stress, $\sigma = P/A$ (5.1)

where P is force acting at the centroid of the section, and A the area of the section.

Tensile stress (note section is in equilibrium)



$\sigma_{tensile} = P/A$

Compressive stress (note section is in equilibrium)

$\sigma_{compressive} = -P/A$

Fig. 5.1 Definition of Direct Stresses

The notion of stress is purely conceptual. What is observed in nature is the deformation of the material. Consider a bar under tension - the length of the bar increases from original length L, to L+ΔL when the force is applied.

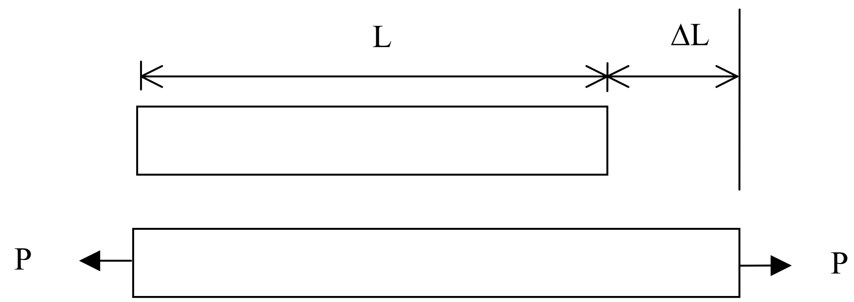


Fig. 5.2 Elongation or Extension of a Bar Under Tensile Load

The axial strain, ε = change in length / original length = $\Delta L/L$ (5.2)

The change in length is called the **extension** or **elongation**. Note that it is possible for a material to experience strain without stress (e.g. unrestrained thermal expansion or contraction), but a material experiencing a stress will experience a corresponding strain. Note also that strain is dimensionless. It is typical to measure strain in terms of micro-strain, or $\mu\varepsilon$; 1 $\mu\varepsilon = 10^{-6}$ so a strain of 0.00036578 is written as 365.78 $\mu\varepsilon$.

b. <u>The Constitutive Relation of Ties/Struts:</u>

As a tie or strut under an external load experiences internal stresses hence strains, it is useful to relate the stress to the strain. A "**constitutive relation**" relates the stress to the strain in a material. For many materials, evidence exists that stress is directly proportional to strain but each material has a different constant of proportionality.

Hence $\sigma = E\,\varepsilon$ (5.3)

E, the constant of proportionality, is called the **Young's Modulus** and has units of stress.

The following table shows E values for some common engineering materials.

MATERIAL	E (GPa)
Steels	190 to 200
Copper	110 to 120
Aluminium	69 to 70
Brass	100 to 120
Glass	50 to 70

Substituting (5.1), (5.2) into (5.3) enables us to derive an expression for the extension of a tie or rod under a tensile force, P. Hence if x denotes the extension,

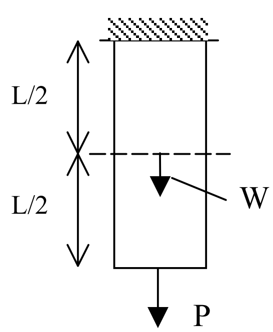
$P/A = E(x/L)$

$$x = PL/(AE) \qquad (5.4)$$

Equation 5.4 is the constitutive relation for ties/struts.

IF #35 The constitutive relation of ties and struts is x = PL/(AE).

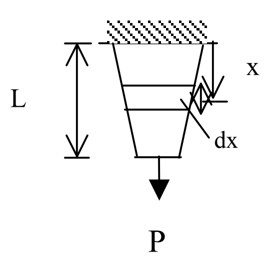
1. Extension of a Bar Under a Load and Including Self-Weight



The self-weight W acts at the center of gravity of the rod which is L/2 from the top, so the extension due to W is WL/(2AE). Hence the extension due to both W and P

$$= WL/(2AE) + PL/(AE) \qquad (5.5)$$

2. Extension of a Tapered Bar



The bar has a radius r_o at the top, r_1 at the bottom, and r at a section x from the top, which is at the middle of an infinitesimal length , dx.

$r = r_o - (r_o - r_1)\ x/L$

If the infinitesimal length extends an amount du due to the load P then from equation 5.4,

$du/dx = P/(A_x E)$ where $A_x = \pi(r_o - (r_o - r_1)\ x/L)^2$

The total extension in the bar, $u = (P/\pi E)\int dx/[\pi(r_o - (r_o - r_1)x/L)^2]$, with the integration between limits of 0,L. Hence,

$$u = (PL/\pi E r_o r_1) \tag{5.6}$$

c. <u>Axial Stress-Strain Behaviour:</u>

In our task as civil engineers of shaping the built environment, it is necessary to use materials in ways that place significant stress on them. It is therefore of vital importance that civil engineers be aware of the behaviour of these materials when loads are applied.

Of special concern is the behaviour of the material when a tensile load is applied. This is because tension causes cracking in the most cost effective building materials - concrete, and masonry, which typically occupy 95 percent of all the materials used in a building or non-building structure. The weakness of concrete and masonry is overcome by using them in conjunction with metals such as steel, which is strong in tension. The behaviour of steel, the most commonly used construction material, in tension must be well understood if it is to be used effectively. Also, the behaviour of steel in tension displays a number of phases enabling us to gain insight into tensile behaviour in general.

Consider the following graph of the typical stress-strain behaviour of a steel rod.

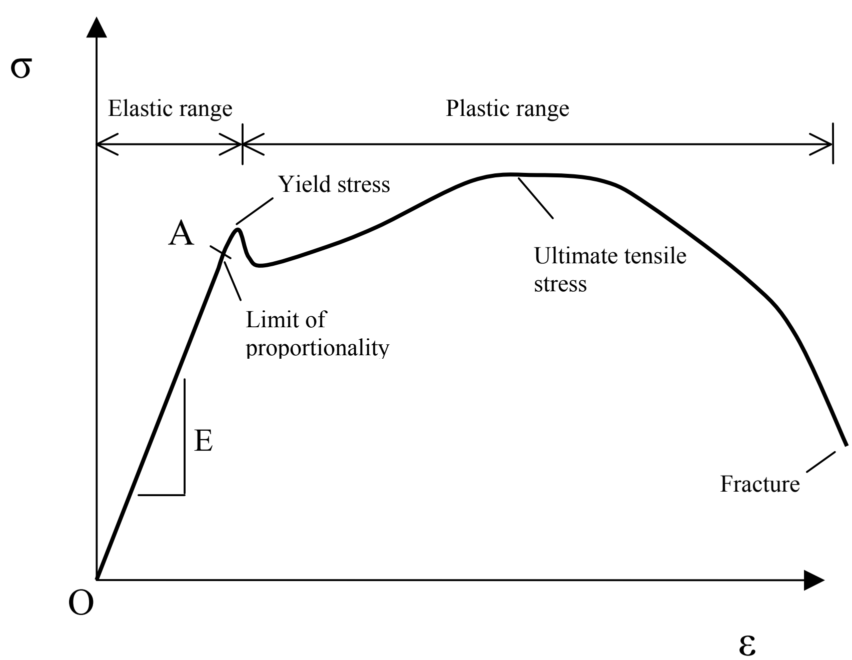


Fig. 5.3

If a steel rod is tested under a tensile load from zero to fracture, and the data points are plotted, the result is similar to the curve shown in Fig. 5.3. There are 3 distinct phases - a linear portion of the curve from O to A followed by a small "hill" region; a gradual rise to a plateau region, then a downward slope to fracture.

Point A at the end of the linear portion is called the **limit of proportionality**. The peak of the little hill is called the **yield point** and the corresponding stress and strain called the **yield stress** and **yield strain** respectively. The subsequent gradual rise is called **strain hardening**, and the maximum stress in the plateau region is called the **ultimate tensile stress**.

The mechanical properties of interest are the Young's Modulus (previously discussed) which is the gradient of the linear portion, the yield stress, σ_y, and the ultimate tensile stress σ_u.

When the load is within the linear range, if the load is reduced back to zero, the elongation also reverses along the original curve to zero value at zero load. Such behaviour is termed "linear elastic". However, when the load is beyond the linear range, if the load is reduced back to zero, the elongation also reverses but not along the original curve and not to a zero value at zero load. This extra strain or elongation is called **permanent set** and indicates that the material experienced **plastic deformation**. Hence the zone of behaviour beyond the yield point is called the **plastic range**.

The length of the curve in the plastic range indicates the **ductility** of the material. This is the most important property of a material for giving earthquake resistance to a structure. After the test, the **percent elongation** of the rod, and the **percent reduction of area** of the section where fracture takes place, also indicate the ductility of metals. For the latter, when the material is close to fracturing, the diameter of the rod near the section where fracture occurs continually reduces. This is called **necking**. Clearly steel has excellent ductility. The shorter the plastic range the less the ductility and the more the **brittleness** of the material making brittleness the converse of ductility. A brittle material has an elastic range only and fractures at the end of that range. Concrete and masonry are brittle and weak in tension unless reinforced with steel.

The area below the curve in the plastic range is called the **toughness** of the material and is a measure of the ability of a structure built with the material to withstand impact type forces.

d. Allowable Stress

The **allowable stress**, also called the **working stress** or **permissible stress**, is the level of stress that we decide we will not subject the material to. In other words, we will use the material in such a way that the applied stress due to the applied loads will be less than the allowable stress.

We limit the applied stress to no more than the allowable stress mainly when we believe that it will be too dangerous to subject the material to higher stresses. The allowable stress is not the stress beyond which the material will fracture. Fracture occurs at the end of the curve as indicated in Fig. 5.3.

Examining Fig. 5.3 again, we notice that the peak or maximum stress that the rod can resist is the ultimate tensile stress, which remains constant over a range of strain. This peak resistance is also called the **strength** of the material. The strength is relative to the type of load applied so we are referring here to the tensile strength. For many steels, the difference between the yield and ultimate stresses is of the order to 25 to 30 percent, so for calculation purposes it is common to take the value of the ultimate stress as being the same as the yield stress. So we generally consider the yield stress as the tensile strength of the steel though we know that this is conservative.

Prior to around the 1950s, the allowable stress was determined by dividing the tensile strength by a constant value called the **factor of safety**, which is in the range of 1.7 to 2.0 depending on the type of metal. The factor of safety is meant to cater to unforeseen but dangerous situations such as poor manufacturing of the steel. Over the years however, it was felt that this was too uneconomical since using a value so far from the strength means that more material will be used to get the job done than is necessary. Therefore nowadays, the tensile strength (i.e. yield stress) is used directly in calculations. To cater for manufacturing errors, the statistics of the strength of the manufactured rods is used to provide a value which in effect functions like a factor safety but is around 1.15, which is significantly smaller than the 1.7 to 2.0, resulting in more economical usage of the material.

e. <u>Direct Stress in Statically Determinate Systems</u>

Recall from Chapter 2 that for a statically determinate system, to determine the unknown forces only the equilibrium equations are required.

In this section we examine situations regarding the calculation of stresses in ties/struts which require consideration of the equations of equilibrium only.

1. Stress in a Rod Considering Self-Weight

Consider a rod of weight w per unit volume and length L, under a load P as shown. The aim is to determine the stress at any section, σ_x

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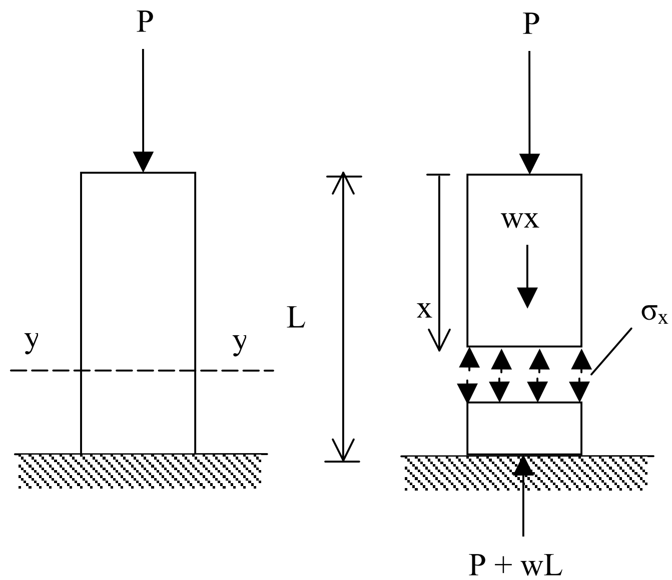


Fig. 5.4

Take a section y-y as shown. We get 2 pieces in equilibrium under the stresses shown to the right.

The equilibrium equation is developed from the fact that the sum of the axial forces must equal zero, which when we divide by A, means that the sum of the stresses must be zero. Taking the top piece and a sign convention of downwards is positive, we therefore get:

$P/A + wx - \sigma_x = 0$

$\sigma_x = P/A + wx$ (5.7)

Equation 5.5 indicates that the stress in the member increases with x and becomes a maximum when x=L at the bottom of the rod.

2. Profile of a Constant Stress Member

In the problem above, the stress depends on the location of the section. Since the stress depends on the cross-sectional area A, it is possible to determine how A must change along the length of the member (i.e. the profile of the member) for the member to have the same stress at any section. Clearly, as x increases we expect A to also increase to balance the increase in the stress due to the increased self-weight. The aim is to determine the profile. Since equation 5.7 indicates that the stress increases linearly we may think that the area must also increase linearly, but this will be incorrect as we now show.

If we consider a section below y-y in Fig. 5.4, the force on that section would increase relative to the force at y-y because of the increased weight of material between the sections. If A is the area of the section at y-y, and A+dA the area of the section below y-y we therefore get:

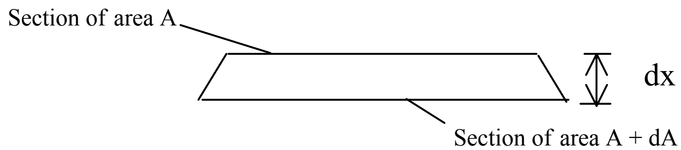


Fig. 5.5 Relationship between successive sections

$\sigma_{x,}$ at A+dA x (A + dA) = $\sigma_{x,}$ at A x A + (w. A . dx) (5.8)

where dx is the length of the member between the 2 sections. However, we desire that the stress at both sections be the same so $\sigma_{x,}$ at A+dA = $\sigma_{x,}$ at A

From equation 5.8,

σ_x dA = wA dx

dA/A = (w/ σ_x) dx (5.9)

Integrating both sides of equation (5.9),

$$\int_{A1}^{A2} dA/A = \int_0^x (w/\sigma_x)dx$$

where A2 and A1 are the area at the lower section, and the area at the top of the member respectiely. This results in,

ln(A2/A1) = (w/ σ_x) x

Hence A2 = A1 $e^{(w/\sigma)x}$

The area at the top, A1 = P/ σ_x , so

A2 = (P/ σ_x) $e^{(w/\sigma)x}$ (5.10)

Equation 5.10 indicates that the profile must decrease exponentially with increasing height if the stress at all sections of a member under a compressive load is to remain the same, and considering the self-weight of the member.

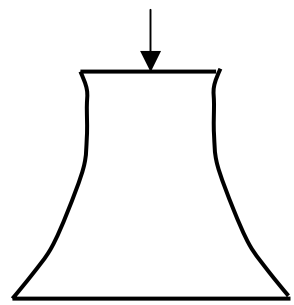


Fig. 5.6 Profile of a Constant Stress Member with Self-Weight Considered

Also, the required area at the base of the member is given by substituting L for x in equation (5.10).

3. Thin-Walled Cylinder Under Pressure

Another example of a statically determinate stress system relevant to civil engineering is the case of pipes under an internal pressure exerted by the fluid in the pipe. This is typical of water supply conduits, gas supply lines, etc. The aim is to determine the stresses given the pressure and properties of the pipe material. Consider the following free body diagram of the stresses.

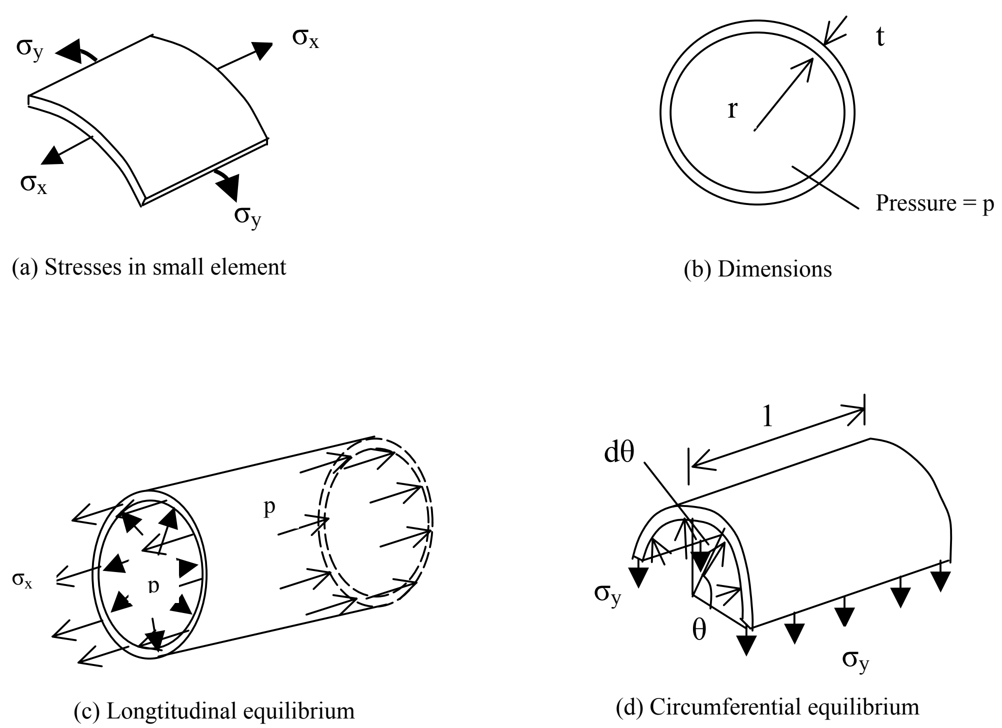


Fig. 5.7

Longtitudinal Equilibrium:

For equilibrium in the longtitudinal direction, the axial force due to the pressure must equal the force in the material. Since the pressure, p, is the same in all directions, and the longtitudinal stress, σ_x acts on the pipe thickness as shown in Fig. 5.7(b),

$$\pi r^2 p = 2 \pi r t \sigma_x$$

Hence,

$$\sigma_x = (pr/2t) \tag{5.11}$$

Circumferential Equilibrium:

To determine the relationship between the pressure and the circumferential stress (also called the **hoop stress**), consider a piece of the pipe formed by cutting longitudinally a length of unity, and cutting transversely through the diameter as shown in Fig. 5.5 (d).

At the cut diameter section, the vertical force in the pipe section must balance the sum of the vertical components of the pressure over the half-cylinder.

$$\text{Vertical force in pipe section} = 2 \sigma_y t \tag{5.12}$$

Since the pressure acts radially across the cross section, the vertical component of the force exerted by the pressure over a small angle of the circumference is $p\ r\ d\theta \sin\theta$.

Hence for θ from 0 to π, total vertical component of the force exerted by the pressure =

$$\int p\ r \sin\theta\ d\theta = 2pr \tag{5.13}$$

Equating (5.13) and (5.12),

$$\sigma_y = p\ r / t \tag{5.14}$$

Comparing with the longtitudinal stress, it is seen that the hoop stress is more critical since it is twice the longtitudinal stress.

Pipe lengths need to be joined since the required length is typically more than the manufactured length. Furthermore, many pipes are made by bending a rectangle and welding along the length. Such a joint can be torn apart by the hoop stress. To cater for this a joint efficiency factor is sometimes used in calculations as a way of reducing the allowable stress.

Allowable stress with a joint present = joint efficiency factor x usual allowable stress

A joint efficiency factor can also be used for end joints (i.e. joining the circumferences of 2 pipes) and in this case it is the longtitudinal stress in the pipe, σ_x, that is of concern.

f. Direct Stress in Statically Indeterminate Systems:

The former problems of this chapter are statically determinate since to determine the unknown stresses only the equilibrium equations are required. In the present section this is not the case. To solve for the stresses require, in addition to the equilibrium equations, that we obtain the additional equations by examining the deformation of the members, and making use of the constitutive relation of equation (5.4) to convert deformation to stress.

1. Stress in a Composite (Compound) Bar:

A tie or strut can be made in such a way that the cross-section is composed of different materials. Such a member is called a **composite** or compound member. The member can be fabricated by either placing the materials concentrically within each other, or symmetrically side-by-side; in both cases the materials need not be bonded together. The important thing is that at the ends of the member there is a rigid connection so each part comprising a different material, moves longtitudinally by the same amount as the other parts.

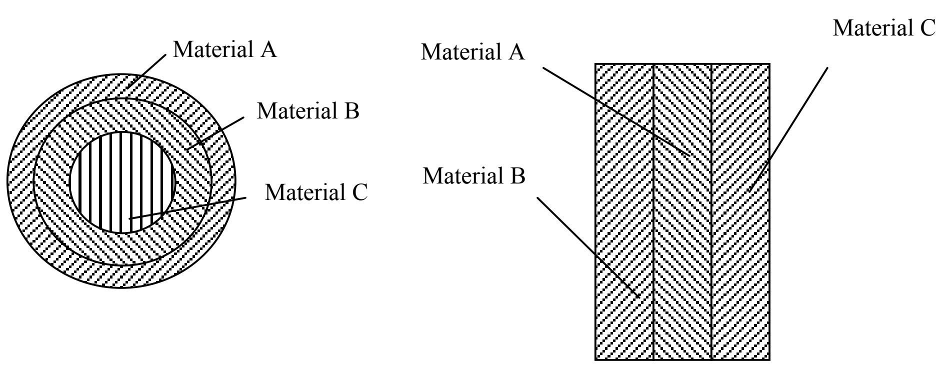
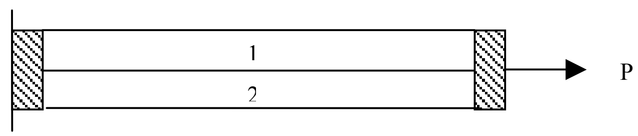


Fig. 5.8 Examples of Composite Cross-Sections

The aim is to determine the stress in each part for a given load or circumstance. As there is only one equilibrium equation, but several materials hence unknown stresses, clearly the problem is statically indeterminate.

1.1 Without Temperature Effects

Consider a composite rod of 2 materials as shown below.



Equilibrium:

$P_1 + P_2 = P$ (5.15)

Consistency of deformation: Extensions must be equal

$u_1 = u_2$ (5.16)

From equation (5.4), equation 5.16 becomes

$P_1/(A_1E_1) = P_2/(A_2E_2)$ (5.17)

From this we get the important fact that,

$\sigma_2 = \sigma_1 (E_2/E_1)$ (5.18)

From eq 5.17,

$P_1 = P_2\, A_1E_1/(A_2E_2)$ but from eq 5.15, $P_2 = P - P_1$ hence,

$P_1 = (P - P_1)\, A_1E_1/(A_2E_2)$

Simplifying we get,

$P_1 = PA_1E_1/(A_1E_1 + A_2E_2)$ (5.19)

Likewise,

$P_2 = PA_2E_2/(A_1E_1 + A_2E_2)$ (5.20)

IF #36 In a composite bar of 2 materials resisting an external axial load P, the internal load on material 1: $P_1 = PA_1E_1/(A_1E_1 + A_2E_2)$, and $\sigma_2 = \sigma_1 (E_2/E_1)$.

1.2 With Temperature Effects

Consider the case where there is no externally applied force P but that the assembly is exposed to a temperature rise T, and rigidly restrained at the ends.

A rod will extend in response to a temperature rise, T, by an amount αTL, where L is the original length of the rod, and α is the **coefficient of thermal expansion** (of unit $°C^{-1}$).

A composite rod not carrying an external load and comprised of materials with different α will experience internal stresses if there is a temperature change. If it is a temperature rise the material with the higher α will pull on the one with the lower α with a tensile force P, and the one with the lower α, will push on the one with the higher α with a compressive force P, and both will experience the same final extension. Note also that though the net force on any cross-section is zero, as each material is of different area and modulus, the sum of the stresses will not be zero.

If $\alpha_1 > \alpha_2$:

Total extension of 1 = $\alpha_1 TL - PL/A_1E_1$

Total extension of 2 = $\alpha_2 TL + PL/A_2E_2$

Due to the end restraints, these extensions hence strains are equal. Therefore,

$$\alpha_1 TL - PL/A_1E_1 = \alpha_2 TL + PL/A_2E_2$$

Hence,

$$P = A_1 A_2 E_1 E_2 (\alpha_1 - \alpha_2) T / (A_1E_1 + A_2E_2) \qquad (5.21)$$

Hence the stress in material 1 (compressive) = P/A_1 and in material 2 (tensile) = P/A_2

IF #37 In a composite bar of 2 materials exposed to a temperature rise T and no externally applied load, the internal load on material 1 and 2 are equal but opposite and given by: $P = A_1 A_2 E_1 E_2 (\alpha_1 - \alpha_2) T / (A_1E_1 + A_2E_2)$. For $\alpha_1 > \alpha_2$ the stress in material 1 (compressive) = P/A_1 and in material 2 (tensile) = P/A_2.

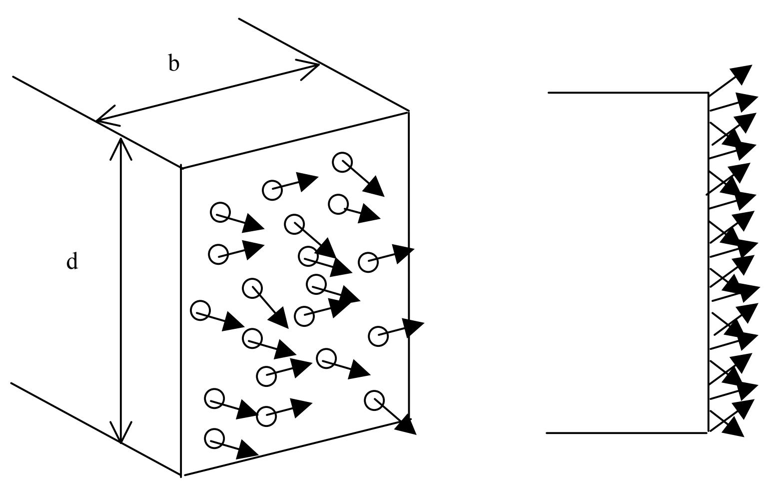
6.0 STRESS, STRAIN IN HOMOGENEOUS BEAM CROSS-SECTIONS (ENGINEER'S BEAM THEORY)

In Chapters 2 and 4 it was explained that bending moments and shear forces develop at any section of a beam in order for the beam to be in equilibrium with the forces at its ends, and the applied forces acting on the length of the beam. These are also referred to as the **applied moment** and **applied shear force** at a section since they are in equilibrium with the applied loads and end forces and are a function of these only.

The applied moment and shear force at a section are resisted, via equilibrium, by the materials of the beam at the section and therefore depend on the properties of those materials and the way they are assembled. If the beam is composed of only one material it is said to be **homogeneous**. If a material is such that there are an infinite number of planes of symmetry through any point, the material is said to be **isotropic**.

b. Shear Force V, and Bending Moment M, as Stress Resultants:

The shear force V and bending moment M at a section are respectively the sum of the vertical components of all stresses at the section, and the sum of the product of all horizontal components and their distances from a horizontal plane at the section. Hence V and M are resultants of the stresses. Consider the following. As shown, there is a stress at each point in the vertical section such that each is in an arbitrary direction.



Isometric view of the typical stresses at a section

Lengthwise view of the typical stresses at a section

Fig. 6.1 Typical Stresses at a Cross-Section

As the stress resultants are summations, they are therefore defined by the following integrals.

$$V_x = \int_A \sigma_y \, dA \qquad (6.1)$$

$$M_x = \int_A \sigma_x \, y dA \qquad (6.2)$$

b. <u>The Constitutive Relation of Beams in Pure Bending:</u>

The aim of this section is to use equation (6.2) to express the internal bending moment at the section, M_x in terms of the properties of the section.

Consider the case of **pure bending**. That is, where over a length of the beam, the bending moment is constant. From equation (1) of Chapter 4, this means that over that length the shear force is zero. Study of the case of pure bending enables us to isolate the stresses that are developed due to bending deformation only.

Consider the following assumptions:

1. Transverse sections of the beam which are plane before bending remain plane after bending.
2. Transverse sections will be perpendicular to circular arcs having a common center of curvature.
3. The radius of curvature of the beam during bending is large compared with the transverse dimensions.
4. Longtitudinal elements of the beam are subjected only to simple tension or compression and there is no lateral stress.
5. The beam is homogeneous and the material isotropic and of Young's modulus the same value in tension and compression.

These assumptions are known to be incorrect with respect to the actual behaviour of the beam. However, relative to the practical concerns of engineers the errors are sufficiently small as to render the resulting equations nevertheless useful. Hence the theory is called the Engineer's Beam Theory. Probably the most significant assumption is the first in which case, the theory is only applicable for beams with a span-to-depth ratio higher than about 25.

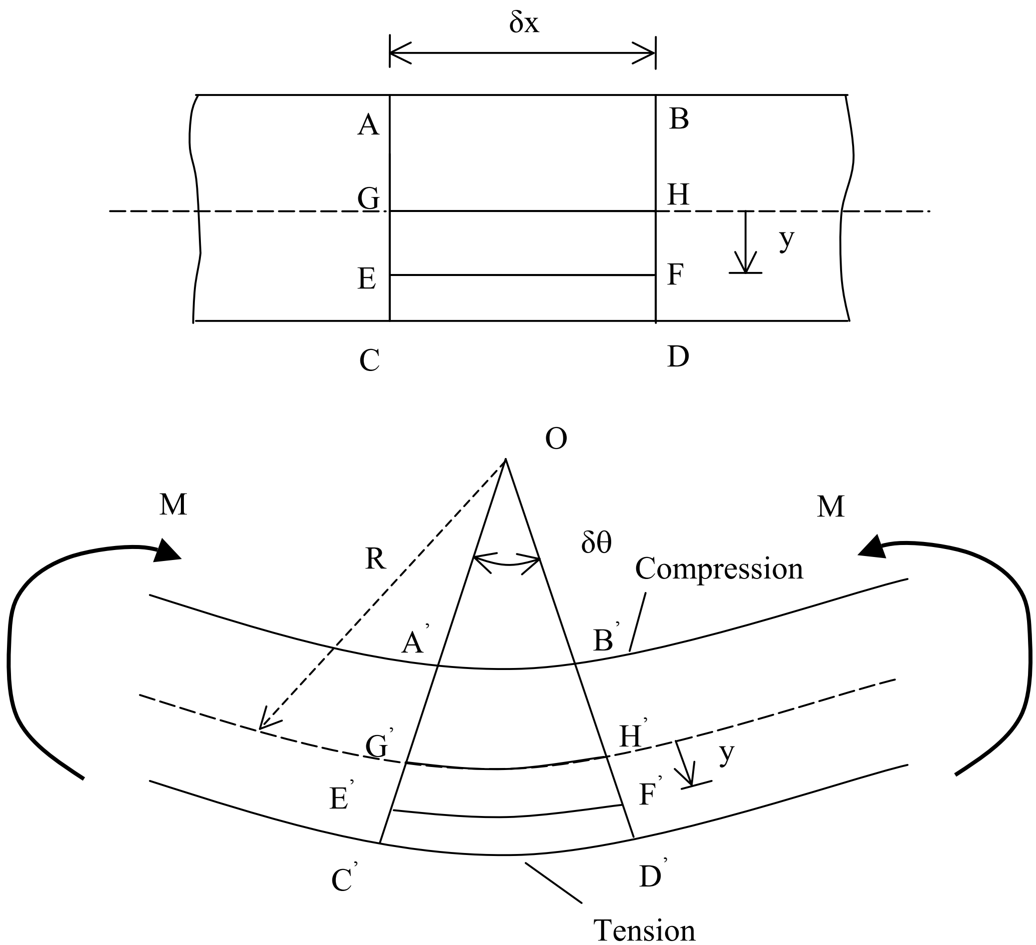


Fig. 6.2 Deformation of a Beam Element Under Pure Bending

The upper part of the diagram in Fig. 6.1 is a portion of the beam before the moment M is applied. The lower part of the diagram is the bending deformation when M is applied. The beam is of arbitrary cross-sectional shape.

The lower surface stretches and is therefore in tension, and the upper surface shortens and is therefore in compression. This implies the existence of a plane where there is zero longtitudinal deformation. This is called the neutral plane and an axis lying in the neutral plane is the neutral axis.

A longtitudinal fiber EF at a distance y below the neutral axis initially has the same length as the fiber GH at the neutral axis. During bending EF stretches to become $E'F'$ but GH being at the neutral axis is unstrained when it becomes $G'H'$. If R is the radius of curvature of $G'H'$,

$G'H' = GH = \delta x = R\ \delta\theta$
$E'F' = (R + y)\ \delta\theta$

Hence the longtitudinal strain in fiber $E'F'$, $\varepsilon_s = (E'F' - EF)/\ EF$

But $EF = GH = G'H' = R\ \delta\theta$

Therefore, $[(R + y)\ \delta\theta - R\ \delta\theta]/\ R\ \delta\theta$

Hence $\varepsilon_s = y/R$ (6.3)

Now assuming a Hookean relationship between stress and strain,

$\varepsilon_s = \sigma_x/E$

hence from equation 6.3, $\varepsilon_s = y/R = \sigma_x/E$

or $\sigma_x/y = E/R$ (6.4)

Substituting for σ_x from equation (6.4), into equation (6.2)

$$M_x = \int_A \sigma_x\ ydA = (E/R)\int_A \sigma_x\ y^2\ dA$$

But from Chapter 1,

$$\int y^2\ dA = I_x = \text{second moment of area, hence}$$

$M_x = E\ I_x/R$ or

$M_x\ /\ I_x = E\ /\ R$ (6.5)

But considering equation 6.4,

$M_x\ /\ I_x = \sigma_x/y = E\ /\ R$ (6.6)

Equation 6.6 is the constitutive relation for a homogeneous beam in pure bending.

Also, since there is no external axial force in pure bending the internal force resultant must be zero.

$$F_x = \int \sigma_x\ dA = 0$$

But from equation 6.4,

$$(E/R) \int y \, dA = 0$$

As E/R is not zero, $\int y \, dA = 0$ but this is the first moment of area, hence the neutral axis must coincide with the centroid of the section.

IF #38 The constitutive relation of a beam is given by M/I = σ/y = E/R.

IF #39 The neutral axis of a beam coincides with the centroid axis of the beam's section.

c. <u>Bending Strain and Bending Stress:</u>

The following are a few points implied by the constitutive relation for a beam that should be explicitly noted.

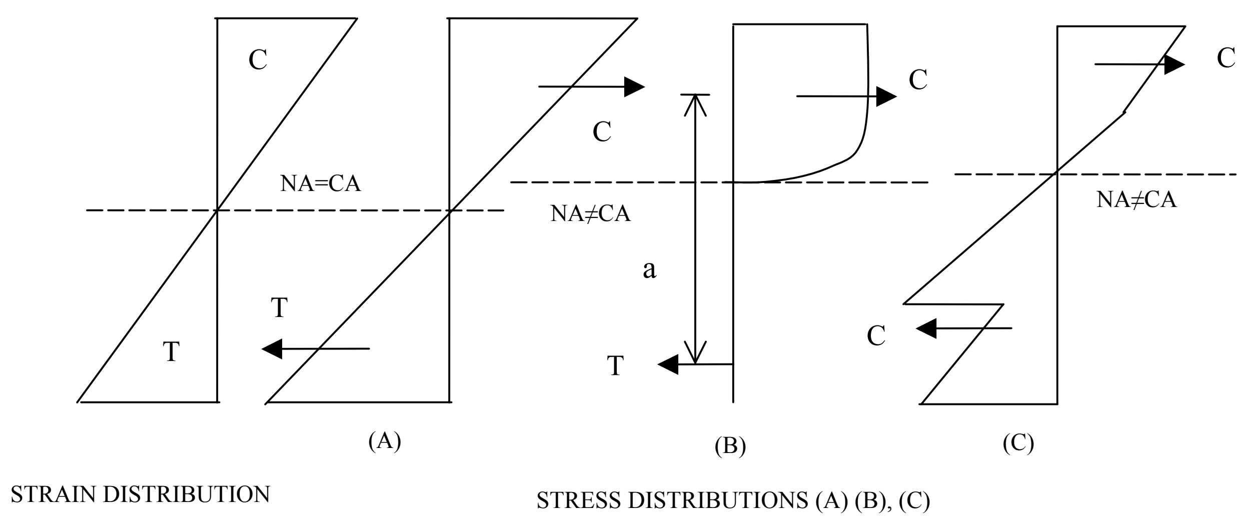


Fig. 6.3 Strain Distribution and Possible Stress Distributions

Fig. 6.3 shows a strain distribution for a beam, and possible stress distributions depending on the material and how it is arranged. Fig. 6.3 (A) is a possible stress distribution if the beam is in the elastic range and composed of a homogeneous material (e.g. steel, timber). Fig. 6.3 (B) is a possible stress distribution for reinforced concrete at failure (i.e. a composite beam in the plastic range), and Fig 6.3 (C) is for a composite beam in the elastic range with unsymmetrical arrangement of the layers.

- The strain distribution is always linear (for the Engineer's Beam Theory) regardless of the beam's material or its cross-section or if the material is in the elastic or plastic range of stress.

- From the strain distribution one can always calculate the stress at any depth of the cross-section.

- For homogeneous beams, or composite beams with a symmetrical arrangement of the layers, the neutral axis (NA) is the same as the centroidal axis (CA) of the section. However, for a composite beam with unsymmetrical arrangement of the layers (e.g. Fig 6.3 (B) and (C)) the NA does not coincide with the CA.

- For horizontal equilibrium C (the total compressive force) and T (the total tensile force) must be equal and opposite. This can be used to determine the position of the neutral axis for non-homogeneous (i.e. composite) beams.

- The most common uses of the constitutive relation for beams are:

  (1) calculation of the (elastic) bending stress at any location along the depth of the beam for a given applied moment M:-

  $\sigma = My/I$ but be know from Chapter 1 that $S = I/y_{max}$, where S is the section modulus.

  Hence the maximum bending stress is given by, $\sigma_{max} = M/S$ (for linear elastic homogeneous materials such as steel and timber)

IF #40 For a given bending moment M, the maximum bending stress in the section is given by $\sigma_{max} = M/S$ where S is the section modulus.

  (2) derivation of the slope and deflection equations for a beam (see Chapter 9)
  (3) the calculation of the maximum moment that a beam can be subjected to (next section).

d. The Moment of Resistance of Beams

In the same way that one can check the safety of a rod by comparing the applied stress to the allowable stress, one can calculate the maximum moment that a section can withstand and compare this with the applied moment.

The maximum moment that a beam section can withstand is called the **moment of resistance** or **moment strength** or **moment capacity**.

Examining Fig. 6.3, notice that T and C form a couple with the lever arm being the distance between them. Hence the moment strength M_n equals this couple and is given by:

$M_n = C$ (or T) x a , which applies to homogeneous and non-homogeneous sections.

For a homogeneous beam of rectangular section and allowable bending stress σ_{all}
$C = T = \sigma_{all}\ bd/4$
$a = d - 2d/(2x3) = 2d/3$
Hence,

$$M_n = (\sigma_{all}\ bd/4) \times 2d/3 = \sigma_{all}\ bd^2/6 = \sigma_{all}\ S \qquad (6.7)$$

IF #41 The moment strength of a homogeneous beam is given by $M_n = \sigma_{all}\ S$ where σ_{all} is the allowable bending stress, and S is the section modulus.

7.0 STRESS IN COMPOSITE BEAM CROSS-SECTIONS

In Chapter 6 the bending stress at a section of a homogeneous beam was presented as well as the means of determining the moment of resistance of a beam given an allowable stress. This has been seen to be quite straightforward and is partly because for such beams the neutral axis coincides with the centroidal axis of the section.

However, recall from Chapter 6 section c, that for a composite beam with unsymmetrical arrangement of the layers the neutral axis does not coincide with the centroidal axis. This gives rise to additional concerns with respect to the determination of the stress distribution at the section, and with the determination of the moment of resistance of a composite beam section. We address these concerns in this chapter for the case where the materials are in the elastic range.

We also consider the case of a section of 2 materials only, though cases of more that 2 materials can be easily determined given the development of the equations for the 2-material case.

a. Composite Beams with Symmetrical Arrangement of the Layers

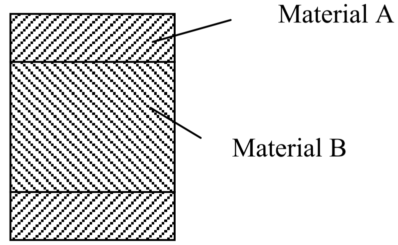


Fig. 7.1 A Symmetrical Composite Beam

For equilibrium, the applied moment M equals the sum of the internal moment contributed by material A, and that contributed by material B. Hence,

$$M = M_A + M_B \qquad (7.1)$$

But from the constitutive relation for a beam that $M/I = \sigma/y = E/R$, this implies that

$$\begin{aligned} M &= E_A I_A/R + E_B I_B/R \\ &= (E_A I_A + E_B I_B)/R \qquad (7.2) \end{aligned}$$

Recall equation 6.3, which applies to a beam of any configuration (i.e. homogeneous or composite). This means that for each fiber of the cross-section the deformation is along a circular arc hence the ratio of strain in the fiber to its distance from the neutral axis is constant for all fibers. Therefore,

$$1/R = \varepsilon_A/y_A = \varepsilon_B/y_B \qquad (7.3)$$

As the materials obey Hook's law,

$\sigma_A = E_A\ \varepsilon_A$ (7.4)
$\sigma_B = E_B\ \varepsilon_B$ (7.5)

Substituting (7.4), (7.5) into (7.3) then (7.2) we get,

$(\sigma_A/\ E_A\ y_A) = (\sigma_B/\ E_B\ y_B) = M\ /\ (E_AI_A + E_BI_B)$

Rearranging we get,

$\sigma_A = M\ E_A\ y_A/(E_AI_A + E_BI_B)$ (7.6)

$\sigma_B = M\ E_B\ y_B/(E_AI_A + E_BI_B)$ (7.7)

Therefore from equations 7.6 and 7.7, one can determine the stress distribution in the composite beam by substituting the values for y at the boundaries of each material.

Note that the sign convention is: positive downward from the centroid of the section; so tensile bending stresses are of positive sign, and compressive bending stresses are of negative sign.

b. <u>Composite Beams with Unsymmetrical Arrangement of the Layers</u>

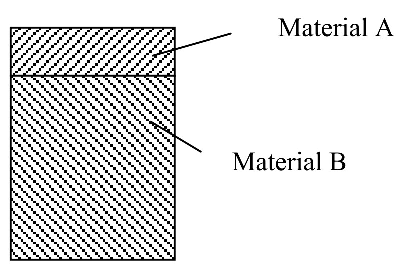


Fig. 7.2 An Unsymmetrical Composite Beam

In the case of the unsymmetrical composite beam, since the neutral axis does not coincide with the centroidal axis, but the stresses depend on the y's which are measured from the neutral axis, this means that we must first determine the location of the neutral axis before using equations 7.6 and 7.7. This is a tedious procedure.

The alternative approach is to **transform the section** into an equivalent section in which the cross-section is composed entirely of one of the materials. Then the neutral axis will coincide with the centroidal axis of the transformed section.

<u>The Transformed Section</u>

Recall the parallel axis theorem of Chapter 1, and IF #6. Then for any part of a section,

$I_x' = I_x + c^2A$

where $I_{x'}$ is the contribution of the part to the I of the whole section, and c is the distance from the centroid of the part to the centroid of the section. For a rectangular section,

$I_x' = bd^3/12 + c^2 bd$

Multiplying by m,

$m I_x' = m (bd^3/12 + c^2 bd)$

$m I_x' = mbd^3/12 + c^2 mbd$

If we make the substitution b' = mb we get

$$m I_x' = b'd^3/12 + c^2 b'd \qquad (7.8)$$

The right-hand side of equation 7.8 is the I for a different or transformed section where the transformation is done by replacing the width of the rectangle, with a new width b' obtained by multiplying the original width b by m.

<u>Stresses at the Interface</u>

Consider the interface of the materials A and B in Fig. 7.2. The strain in each material at the interface must be the same. Hence,

$\varepsilon = \sigma_A / E_A = \sigma_B / E_B$. Rearranging, we get

$$\sigma_A = \sigma_B (E_A / E_B) \qquad (7.9)$$

If $E_A > E_B$ the ratio E_A/ E_B is called the **modular ratio**. Hence,

$$\sigma_A = m\, \sigma_B \qquad (7.10)$$

Moment at the Section

By definition, the moment at a composite section is independent of the arrangement of the layers, hence recalling equation 7.2

$M = (E_A I_A + E_B I_B)/R$

As $E_A/R = \sigma_A/y$ and $E_B/R = \sigma_B/y$

$M = (\sigma_A I_A + \sigma_B I_B) / y$

From equation 7.10 and substituting for σ_A we get,

$$M = \sigma_B (mI_A + I_B)/y \qquad (7.11)$$

Alternatively, if we substitute for σ_B we get,

$$M = \sigma_A (I_A + I_B / m)/y \qquad (7.12)$$

Considering equation 7.11, this is the same as saying that the moment M is obtained relative to an **equivalent section** composed entirely of material B, and of an I given by $(mI_A + I_B)$.

However, given equation 7.8, the mI_A of 7.11 means that in the equivalent section, the width of the zone originally of material A, now has a width mb where b is the width of material A in the original composite section.

Alternatively, if equation 7.12 is being used (i.e. equivalent section entirely of material A), then in the equivalent section, the width of the zone originally of material B, now has a width b/m where b is the width of material B in the original composite section.

Procedure for Calculating the Stresses

Given the aforesaid, the following is the procedure for calculating the bending stresses, hence obtaining the bending stress distribution, for an unsymmetrical composite beam section:

Step 1. Transform the original section, which has 2 materials, to give an equivalent section of one of the materials only. If we call the material with the higher E value as material A, then the equivalent section is entirely of material B and the width of the zone originally occupied by material A is multiplied by m to get the new width of that zone in the transformed section.

Step 2. Given the transformed section from step 1, as it is entirely of one material, the neutral axis now coincides with the centroid of the section. Calculate the position of the centroid and the I_x of the section.

Step 3. Given the centroid of the transformed section determine the y value at the top, bottom and interfaces. Knowing I_x from step 2, calculate the stresses at these locations from $\sigma_B = MI_x /y$

Step 4. Step 3 gives the stress distribution of the transformed section which is entirely of material B. To get the stress distribution for the original composite the stress in material A is now required. This is obtained by using $\sigma_A = m\ \sigma_B$. Note that at any interface between material A and B, there will be a stress value for each, so the distribution at an interface in the composite section will have these 2 values.

This procedure of using the equivalent section to determine the stresses, can also be used for symmetrical sections.

c. <u>The Moment of Resistance of Composite Beams</u>

Recall IF #41 from Chapter 6 that to determine the moment of resistance of a homogeneous beam section you substitute the allowable stress into the constitutive relation, so $M_{res} = \sigma_{all}$ x I/y_{max}.

In the case of a composite beam section however, we have (at least) 2 materials each with its own allowable or permissible stress.

In the last section we express the stresses in the composite beam section in terms of either one of the materials. Therefore, the issue with the calculation of the moment of resistance for a composite beam section is that if we use the allowable stress for one of the materials, we do not immediately know if the stress in the other material would then be higher than its allowable value. This means that for composite beam sections, we must do the calculations twice - one for each material, and the correct answer is the safe one. The safe answer is the one for which the stress in each material is less than or equal to its allowable stress.

For symmetrical sections, we can directly calculate the moment of resistance based on each material so the correct or safe answer is the lower answer. For unsymmetrical sections, we determine the equivalent section for each material and calculate the stresses in the both materials. The safe stress distribution is the one for which the stress in each material is less than or equal to its allowable stress. We then calculate the moment of resistance by using the safe stress distribution and substituting in σ_{all} x I/y_{max} for the safe equivalent section.

Symmetrical Sections

Equations 7.6 and 7.7 can be re-written in terms of the moment of resistance and allowable stress as,

$$M_{res} = (\sigma_{A,\ allowable} / y_{A,\ max}) (I_A + I_B/m) \tag{7.13}$$

$$M_{res} = (\sigma_{B,\ allowable} / y_{B,\ max}) (mI_A + I_B) \tag{7.14}$$

where m is the modular ratio E_A/E_B.

In equation 7.13, $y_{A,\ max}$ is the y value for the extremity of material A that is furthest away from the centroid of the composite section. Note that this value need not be the maximum y value for the section as a whole, but only for material A.

Note also that if the section is such that the layers are horizontal, it is most convenient to use the parallel axis theorem to calculate the I's remembering not to forget the b^2A term.

Upon substitution in equations 7.13 and 7.14, the moment of resistance of the symmetrical section is the lower value.

Unsymmetrical Sections

Since we typically use the transformed section is this case, and the transformed section is of one material, the calculation of the moment of resistance for unsymmetrical sections is the same as for homogeneous sections, after we determine the safe stress distribution.

Step 1. Transform the section into one with material A only and knowing the allowable stress for material A, determine the stress distribution of this section.
Step 2. In the stress distribution of step 1, at the level where there is an interface between materials A and B, calculate the stress at that level for material B using $\sigma_B = \sigma_A/m$.
Step 3. Repeat steps 1 and 2 but this time for material B, and using $\sigma_A = m\sigma_B$.
Step 4. Knowing the stresses in both materials for each case (i.e. case when material A is at its allowable stress from step 1, and the case when material B is at its allowable stress from step 2), determine the safe stress distribution.
Step 5. Knowing the safe stress distribution, calculate M_{res} by substitution in $M_{res} = \sigma_{all} \times I/y_{max}$.

IF #42 Regardless of whether composite beam section is symmetrical or not,
$M = \sigma_B (mI_A + I_B)/y = \sigma_A (I_A + I_B / m)/y$, $m = E_A/E_B > 1$

IF #43 If $E_A > E_B$, to create a transformed section entirely of material B, multiply the width b of the zone occupied by material A, by m to give a new width mb. To create a transformed section entirely of material A, multiply the width b of the zone occupied by material B, by 1/m to give a new width b/m. The new section can now be treated as a homogeneous beam section.

IF #44 At the interface of 2 materials in a composite beam $\sigma_A = m\sigma_B$, if $E_A > E_B$.

8.0 **COMBINED AXIAL AND BENDING STRESS**

Consider a section in the linear elastic range. An axial load applied to the section at the centric of the section will result in the same stress (P/A) at any point in the section, whether that stress is tensile or compressive.

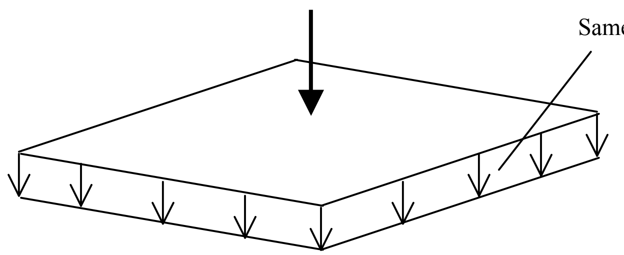


Fig. 8.1 An Axial Load at the Centroid

A bending moment applied about the centroidal axis of the same section (assuming a homogeneous section) will result in bending tensile and compressive stresses in accordance with the theory of bending of beams (My/I).

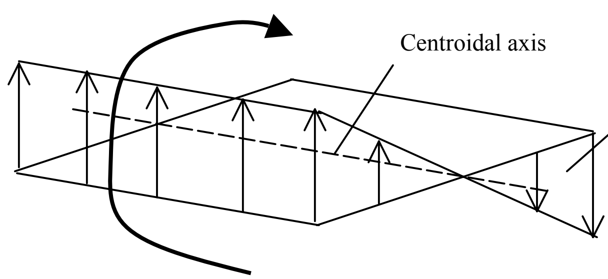


Fig. 8.2 A Bending Moment about the Centroidal Axis

If both the axial load and the bending moment are applied simultaneously, the resulting stress at any point can be determined using the principle of superposition. Hence the resulting stress is simply the sum of the axial stress, and the bending stress at that point.

Note however that when axial and bending stresses are combined, the stress on one side is increased whereas on the other side the stress is decreased. This is because the bending moment induces both tensile and compressive stresses whereas the axial load induces either tensile or compressive stresses.

In the above, the bending moment M, is applied independent of the applied load P and the two of them together create the combined stress state. However, a bending moment is created when the axial load P is applied at a point other than the centroid. The difference between the actual position of P and the centroidal location is called the **eccentricity, e**. This is shown in the figure below for the one-dimensional case. Hence a moment is induced by the eccentric axial load, with the value of the moment as Pe.

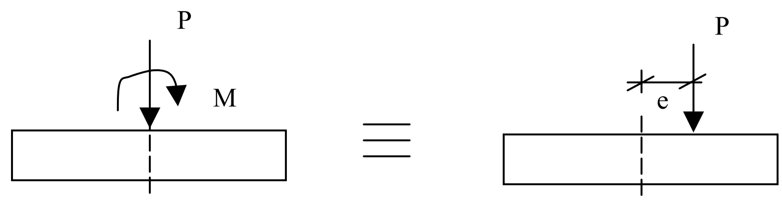


Fig. 8.3 Moment as Equivalent to Eccentric Axial Load

In a situation of combined loads we are concerned with 2 main questions – (a) what is the net or resultant combined stress at any point, and (b) when is the net or resultant combined stress either totally compressive or totally tensile. We consider these in the subsequent sections.

a. Sections with Uniaxial Bending

In the above description of combined stresses, we used a rectangular section to illustrate the point, but the section can be of any shape. There is also only one moment about any of the centroidal axes, so the type of bending indicated is called **uniaxial bending**.

Hence, to be consistent with the sign convention for bending stresses as presented in Chapter 6, if we consider a compressive stress as negative, the resultant combined stress at any point is given by,

$$\sigma = -P/A + My/I \qquad (8.1)$$

In equation 8.1, if the point under consideration is to the right of the centroid (i.e. neutral axis) then y will be negative, and vice versa. Graphically, equation 8.1 means:

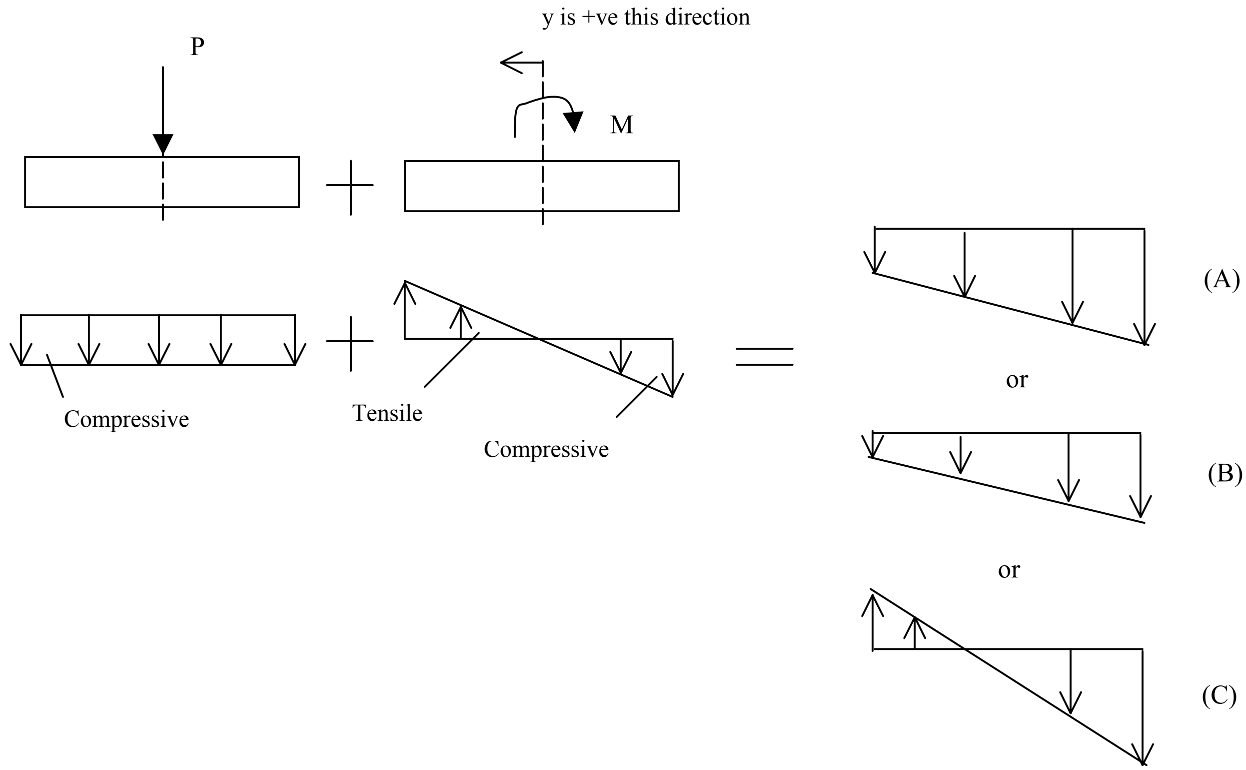
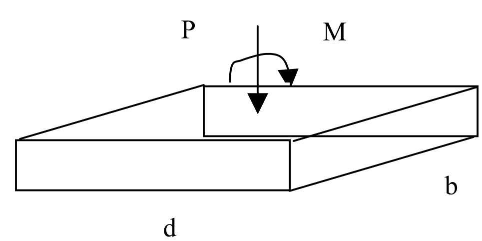


Fig. 8.4 Possible Net Combined Axial and Bending Stresses

The diagrams of Fig. 8.4 can be considered the same as Figs. 8.1 and 8.2 but viewed in the plane of the applied moment, M. Note that there are 3 possibilities with respect to the net combined stress – (A), (B), and (C). These show how the stresses change as the ratio M/P increases. In (A), and (B), the stress is compressive at any point but decreases on going from right to left. Eventually, if M/P is large enough, a zone of tensile stresses develop as shown in (C). This situation is on concern in the design of pad footings in which case it is desirable to have only compressive stresses. Now since M = Pe this is the same as saying that as the eccentricity e increases, a point is reached where tensile stresses can occur.

Let us determine this point for the case of a rectangular section.



$A = bd$ and $I = bd^3/12$

At the point in question, the net compressive stress equals zero and $y = d/2$, hence substitute in equation 8.1 and noting that $M = Pe$, we get,

$0 = -P/(bd) + 6Pe/(bd^2)$

Hence,

$P/(bd) = 6Pe/(bd^2)$. Dividing on both sides by $P/(bd)$ we get

$1 = 6e/d$ or,

$e = d/6$

This tells us that if the ratio of M/P or e, for the sense of the moment shown, is greater than d/6 then tensile stresses will develop, otherwise the section will have net compressive stresses only. Since the moment can be of opposite sense, this means that in a zone of 2x(d/6) centered at the centroid of the section, if P is placed within that zone the section will never develop a net tensile stress at any point. This zone is therefore called the **middle third**.

It is also noteworthy that for the uniaxial bending case, all points on the section distance y from the centroid have the same net combined stress (as shown in Fig. 8.2).

b. Sections with Biaxial Bending

The equations of the last section were developed for the case of uniaxial bending in which case there is only one moment applied to the section and it is about a centroidal axis. In many practical situations however, there is bending simultaneously about both centroidal axes. This type of bending is called **biaxial bending**.

The case of combined axial and biaxial bending stresses follows by extension of the uniaxial case. For the biaxial bending we consider the bending about the centroidal z-z axis as M_z, and about the x-x axis as M_x. We will also have an eccentricity e_x of P relative to the z-z axis, and an eccentricity e_z of P relative to the x-x axis.

Consider a coordinate system as follows established for a plane section of arbitrary shape. This gives the positive directions for all quantities.

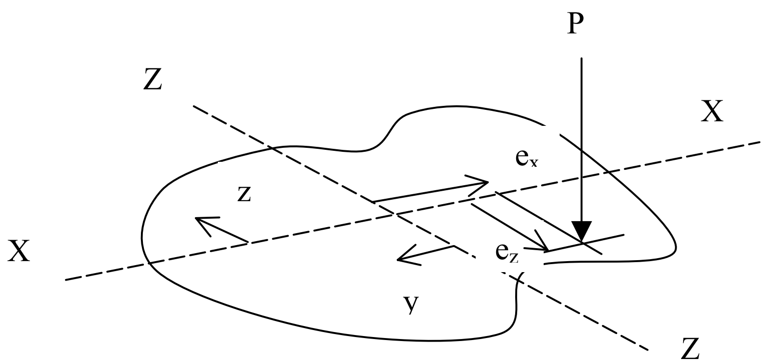


Fig. 8.5 An Arbitrary Plane Shape Under Biaxial Bending

Hence as for equation 8.1, by the principle of superposition we get the following equation for the stress at any point (y,z) on a plane due to combined axial and biaxial bending stresses as,

$$\sigma = -P/A + M_z\,y/I_{zz} + M_x\,z/I_{xx} \tag{8.2}$$

This can also be written as,

$$\sigma = -P/A + P\,e_x\,y/I_{zz} + P\,e_z\,z/I_{xx} \tag{8.3}$$

Using equations 8.2 and 8.3 it can be shown that for a rectangular section, the zone beyond which a net tensile stress will develop in some corner of the section is of a rhombus shape with extremities ±d/6 and ±b/6 from the centroid. This zone is called the **kern**.

Lastly, it is noteworthy that for the case of combined axial and biaxial bending stresses the stresses at each corner of the section will be different.

IF #45 For the case of uniaxial bending, the resultant combined axial and bending stress at any point is given by, $\sigma = -P/A + My/I$.

IF #46 For the case of biaxial bending, the resultant combined axial and bending stress at any point is given by, $\sigma = -P/A + M_z\,y/I_{zz} + M_x\,z/I_{xx}$.