

CE 31B STRUCTURAL ENGINEERING

STRUCTURAL DYNAMICS

1. Fundamental Characteristics of Vibration

Vibration is a time-varying response to a change in motion of a structure in such a manner that inertia forces are imposed which resist the change. Only when inertia forces arise and affect the system, is the phenomenon considered a phenomenon of dynamics.

In the presence of inertia forces, the system is governed by Newton's Second Law such that it is in a state of dynamic equilibrium (d'Alembert's Principle). If the system is disturbed and the disturbance removed, the system tries to return to its former state and vibrates on its own until it does so. This "vibration on its own" is called *free vibration*. Vibration due to an external time-varying disturbance is called *forced vibration*. During free vibration, the structure moves from side to side and the time taken to move from one position, back to that original position, is called the *natural period*. Moving from one position, back to that original position, is called making a *cycle*, so the period, is the time required to make one cycle. Also, the number of cycles made in one second (Hertz, Hz), is the *frequency*.

The nature of dynamic equilibrium is that the system or structure always vibrates as a combination of basic deflected shapes of the structure. These basic shapes are called the *modes of vibration*. To see the modes of vibration of, for example, a guitar string, you pluck the string, then shine a strobe light on the string in such a way that the frequency of the light is the same as the frequency of the particular mode of vibration. The string appears to be still but bent, though in fact it is still vibrating – a *stationary wave*. Change the frequency of the strobe light to match that of the next mode of vibration of the string, and another shape appears. This can go on ad infinitum. The mode shape with the lowest frequency is called the *fundamental mode of vibration*, and that lowest frequency is likewise called the *fundamental frequency*. It is important to note that the mode shape is a stationary wave. The reason it is stationary is because when a system is disturbed it always tries to return to its undisturbed state. This tendency is present in any kind of vibration problem and this is why in solving structural vibration problems, the solution is always found as a combination of the modes of vibration.

This tendency manifests mathematically, in the solution of the dynamic equilibrium equations that govern the motion, as the solution to the well-known eigenvalue problem. Hence the modes of vibration are eigenvectors, and the modal frequencies are the corresponding eigenvalues. Therefore solving the attendant eigen-problem is a sub-set of any total solution of a vibration problem (except for the simplest of cases).

To complete the review of basic vibration phenomena, it must be mentioned that free vibration is always accompanied by *damping*. This is the decay in the free vibration over time due to friction. If there is too much damping, the structure will not oscillate and is called *over-damped*. The amount of damping that just causes over-damping is called the *critical damping*.

2. Uses of Structural Dynamics

Building and civil engineering facilities are comprised of structures. The main areas in which structural dynamics is applied are - earthquake engineering of building structural systems; earthquake engineering of building non-structural systems (e.g. machinery; suspended pipes, etc.); earthquake engineering of soils; earthquake engineering of infra-structural elements (e.g. bridges; culverts; tanks; etc.); earthquake engineering of lifeline systems (e.g. embedded pipes, etc.); wind engineering of buildings, transmission towers, etc.; blast engineering of structures, and coastal engineering in terms of the effects of waves on shore line and off-shore facilities.

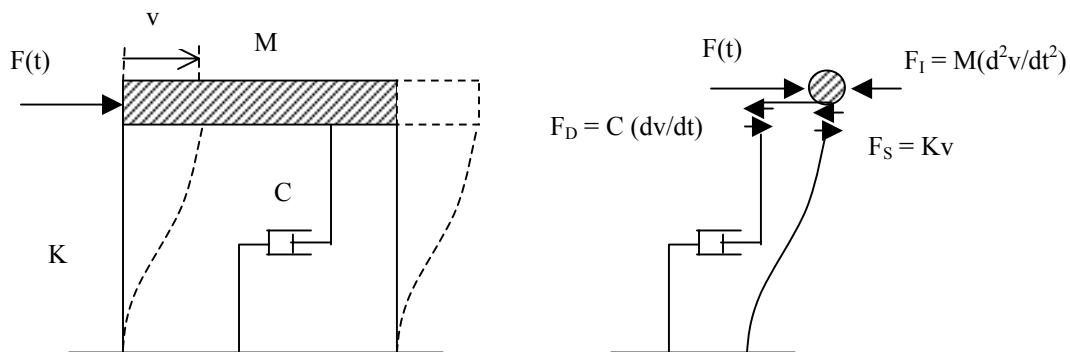
This presentation focuses on the area of earthquake engineering of building structural systems although the fundamental approaches and solutions apply to many of the other areas as well.

In this area, the following dynamic analysis approaches are used, in descending order of regularity. For this course, only the main aspects of modal analysis, and the MDOF Response Spectrum approaches are presented.

APPROACH	DESCRIPTION	LIMITATION	COMMENTS
1. SDOF Response Spectrum - ELF Procedure	The basic approach used in design codes; no time variation	For regular structures only; requires the use of a correlation variable, R_w , for typical earthquake design.	This is a special case of the MDOF Response Spectrum approach
2. MDOF Response Spectrum	A generalised version of the above.	For irregular structures; requires the use of a correlation variable, R_w	Derived from modal analysis but by focusing on peak values; very useful for a wide range of building design problems
3. Modal Analysis Time History	Uses the property of orthogonality to separate the motion into a set of independent motions	Linear. Analytical solutions only available for simple forcing functions but not earthquake excitation.	Provides results for any time, t , during the disturbance.
4. Direct Integration - Linear Elastic	The governing equations involve linear constitutive relations and are solved by numerical integration	Linear, but can solve for more complex forcing functions	Can be used with modal analysis. Also provides results for any time, t , during the disturbance.
5. Direct Integration - Non-linear	The governing equations are non-linear since they require non-linear constitutive relations; solved by numerical integration	More computer intensive since it requires an optimisation solver	Most accurate of the deterministic approaches; rapidly gaining prominence for performance-based design due to improved computer power/cost.
6. Probabilistic	The governing equations are probabilistic	The reliability estimates are not absolute but relative hence can only be used for comparing designs	Can be linear or non-linear. Used by researchers and code developers.

3.0 The Horizontal Vibration of a SDOF Structure

A SDOF system is simply a representation of the structure as a single lumped-mass that can only move one way. Hence it is called a single degree of freedom (SDOF) system. A building modelled in this way where the mass moves sideways is called a shear frame. This system embodies most of the concepts associated with the engineering mathematics of vibration. It is also a necessary introduction to modal analysis since modal analysis works by separating the structure into a set of separate SDOF's.



According to d'Alembert's principle,

$$F_I + F_D + F_S = F(t)$$

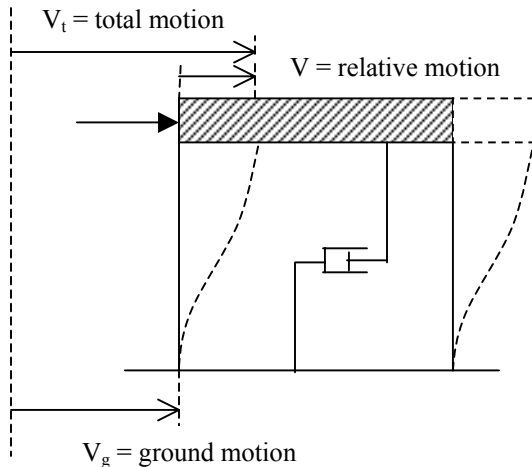
Where F_I = inertial force

F_D = damping force

F_S = force resisted by spring

Hence,

$$M(d^2v/dt^2) + C (dv/dt) + Kv = F(t) \quad (1)$$



For ground acceleration, the inertia force on the lumped mass is with respect to the total motion such that,

$$F_I = M(d^2v/dt^2 + d^2v_g/dt^2)$$

Therefore with respect to the forces on the lumped mass and hence in terms of the relative motion of the structure, eq(1) becomes,

$$M(d^2v/dt^2 + d^2v_g/dt^2) + C (dv/dt) + Kv = 0, \text{ or}$$

$$M(d^2v/dt^2) + C (dv/dt) + Kv = -Md^2v_g/dt^2 \quad (2)$$

Eq (2) is the governing equation of a structure under ground acceleration modelled as a SDOF shear building.

It is one of the most fundamental governing equations of structural dynamics. Now v and v_g are functions of time. Therefore, the various kinds of structural SDOF dynamics problems encountered are simply due to the definition of v_g as it varies with time:

When,

$v_g = 0$, we get the free body problem

$v_g =$ a simple harmonic function such as $v_0 \sin \omega t$, we get the classic equations of damped vibration which mathematically displays the central features of resonance, over-damping etc.

$v_g =$ any general function, we get the general solution of any damped vibration problem. This solution is called the Duhamel or convolution integral. The Duhamel integral either can or cannot be solved in closed-form (i.e. analytically), in which case numerical integration is used.

The solution of these cases follow.

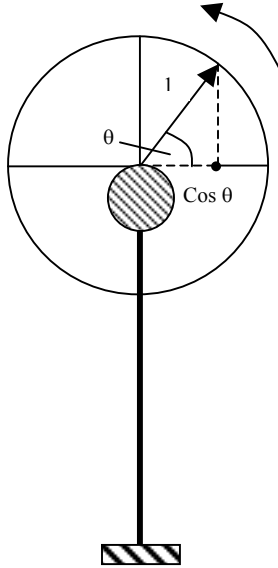
3.1 Free Vibration - Transient Ground Acceleration

The free vibration problem is also called the transient response problem. This is because just after $t=0$, when the ground acceleration begins, regardless of the type of ground acceleration, this initial value is like an impulse acceleration which the system responds to as a free vibration superimposed on the subsequent ground acceleration. The effect gradually fades so is called the transient response. After fading away, the response is called the steady-state response.

If no ground motion is applied to an SDOF system without damping, eq(2) is simplified to,

$$M(d^2v/dt^2) + Kv = 0 \quad (3)$$

Simpler solutions tend to be obtained if trigonometric functions are used to solve (3) for the displacement v . Since the displacement of the mass occurs along a straight line, the unit circle concept shows how this is achieved.



Consider the unit circle with the radius rotating anti-clockwise ω times per second. The displacement is the horizontal projection of the unit radius. The projection follows the radius as it rotates such that when the radius is at 0 deg, the projection length is the same as the radius (=1), and when the radius is at 90 deg, the projection length is zero. In other words, as the unit radius rotates, the end of the projection length (the dot) vibrates.

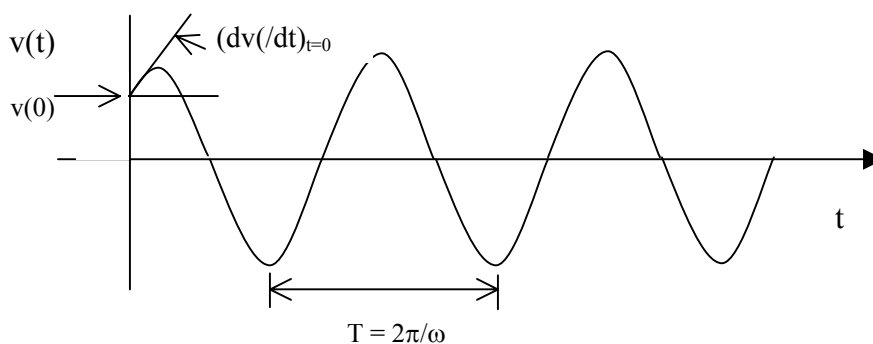
Since the unit radius makes ω revolutions per second this implies that the radius rotates ωt times in t seconds. But for one revolution θ is 2π and the time taken is the period, T . Hence,

$$\omega T = 2\pi$$

ω is therefore called the *circular frequency* (also the frequency of vibration of the dot).

Substituting the harmonic relation, $v = A \cos \omega t + B \sin \omega t$, shows that this is the general solution to (3). From the trig identity $A \cos \theta - B \sin \theta = C \cos(\theta + \alpha)$, it can be shown that the peak value or amplitude of v is $(A^2 + B^2)^{1/2}$. Also, $\omega = \sqrt{K/M}$, hence $T = 2\pi/\omega = 2\pi (M/K)^{1/2}$.

The constants A and B are determined from the initial conditions. At $t = 0$, the constants A and B are $v(0)$, $(dv/dt)_{t=0}$, respectively.



If viscous damping is present in the system, (3) becomes,

$$M(d^2v/dt^2) + C (dv/dt) + Kv = 0 \quad (4)$$

Dividing by M , we get,

$$d^2v/dt^2 + 2\xi\omega(dv/dt) + \omega^2v = 0 \quad (5)$$

where $2\xi\omega = C/M$

The solution of (5) is,

$$v = A \exp(\lambda_1 t) + B \exp(\lambda_2 t)$$

$$\text{where } \lambda_1, \lambda_2 = \omega [-\xi \pm (\xi^2 - 1)^{1/2}] \quad (6)$$

(6) indicates that the solution changes form according to the value of ξ .

If $\xi^2 < 1$,

$$v = C \exp(-\xi\omega t) \sin(\omega_D t + \theta) \quad (7)$$

$$\text{where } C = (A^2 + B^2)^{1/2} \quad \theta = \tan^{-1} A/B$$

$$\omega_D = (1 - \xi^2)^{1/2} \omega \quad (8)$$

For initial conditions of $v = 0$ and $(dv/dt)_{t=0}$ at $t=0$, (7) becomes

$$v = [(dv/dt)_{t=0}/\omega_D] \exp(-\xi\omega t) \sin \omega_D t \quad (9)$$

ω_D is called that the *damped vibration frequency*. The portion of the equation before the sine of (7) and (9), indicates that the system experiences a decaying oscillation with t .

If $\xi^2 > 1$, the system does not oscillate since the effect of the damping overshadows the oscillation (over-damping).

The condition $\xi^2 = 1$ indicates a limiting value of damping at which the system loses its vibratory characteristics; this is called *critical damping*. From (5), the critical damping constant,

$$C_{cr} = 2\omega M = 2(MK)^{1/2} \quad (10)$$

ξ is defined in terms of C_{cr} as, $\xi = C/C_{cr}$. Hence ξ is called the *fraction of critical damping*, or the *damping ratio*. Typical values of ξ are 5% for reinforced concrete, and 2% for steel.

From (7) we get a means for experimentally determining ξ . It can be shown that for successive vibration amplitudes of a damped system,

$$\ln(v_n/v_{n+1}) \approx 2\pi\xi. \text{ This is called the } \textit{logarithmic decrement}.$$

3.2 Forced Vibration - Steady-State Ground Acceleration

If the system is subjected to a sinusoidal ground motion, then after the aforementioned transient response has dissipated, the steady-state response is obtained.

$$v_g = \alpha_0 \sin \omega' t, \text{ where } \omega' \text{ is the frequency of the forcing function,}$$

Hence,

$$d^2v/dt^2 + 2\xi\omega(dv/dt) + \omega^2v = -\alpha_0 \sin \omega' t \quad (11)$$

The solution of (11) is the sum of the complimentary solution (7), and the particular solution. Owing to the complimentary solution, which is the damped free vibration problem, it can be said that the system is always trying to get back to the undisturbed or free state. The particular solution is,

$$v = C_1 \sin \omega' t + C_2 \cos \omega' t \quad (12)$$

Substituting (12) into (11),

$$v = -(\alpha_0/\omega^2) [(1 - \beta^2)^2 + 4\xi^2\beta^2]^{-1/2} \sin(\omega' t - \theta) \quad (13)$$

$$\text{where } \theta = \tan^{-1} (2\xi\beta/(1 - \beta^2))$$

$$\beta = \omega' / \omega$$

It is instructive to investigate the effect of β on the increase in the load, called the *amplification*, due to the influence of the inertia.

A static external force equal to the inertial force ($M\alpha_0$) makes the system deform by $M\alpha_0/K = \alpha_0/\omega^2$. This deflection, labeled v_{st} is,

$$v_{st} = \alpha_0/\omega^2$$

Hence according to (13) the ratio of the dynamic to static deflection, called the *dynamic displacement amplification factor*, is

$$D_d = |v/v_{st}| = [(1 - \beta^2)^2 + 4\xi^2\beta^2]^{-1/2}$$

When D_d is plotted as a function of the frequency ratio β , and for different ξ , we get Appendix Fig 1.

Note that as ω' approaches ω the amplitude increases; this tendency is greater for smaller values of ξ . The condition at which $\omega' = \omega$, or $\beta = 1$, is called *resonance*.

It was stated earlier that the absolute acceleration is $d^2v/dt^2 + d^2v_g/dt^2$, hence from (2),

$$d^2v/dt^2 + d^2v_g/dt^2 = -(\omega^2 v + 2\xi\omega (dv/dt)) \quad (14)$$

Substituting (13) and its first derivative into (14) we get

$$d^2v/dt^2 + d^2v_g/dt^2 = \alpha_0[(1 - \beta^2)^2 + 4\xi^2\beta^2]^{-1/2} [1 + 4\xi^2\beta^2]^{1/2} \sin[\omega' t - (\theta - \theta_0)] \quad (15)$$

where $\theta_0 = \tan^{-1} 2\xi\beta$. Thus the ratio of the response acceleration to the ground acceleration is,

$$D_a = |(d^2v/dt^2 + d^2v_g/dt^2)/d^2v_g/dt^2| = [(1 - \beta^2)^2 + 4\xi^2\beta^2]^{-1/2} [1 + 4\xi^2\beta^2]^{1/2} \quad (16)$$

D_a is called the *dynamic acceleration-magnification factor* and when D_a is plotted as a function of the frequency ratio β , and for different ξ , we get Appendix Fig 2. At $\beta = \sqrt{2}$ D_a is unity regardless of the values of ξ , and D_a becomes smaller than unity in the range $\beta > \sqrt{2}$.

Forced Vibration - Non-Steady-State Ground Acceleration

For the case where the ground acceleration is an arbitrary general function, as is the case in real earthquakes, the solution is obtained by a technique in which the ground motion is considered to correspond to the sum of a series of impulsive loads. However, an impulsive load problem is the same as a free vibration problem but with different initial conditions.

The effective external force caused by arbitrary ground motion is,

$$F(t) = -Md^2v_g/dt^2 \quad (17)$$

Taking $F(t)$ as an impulsive load applied during an infinitesimal time interval $d\tau$, and from the condition that the momentum Mdv/dt equals the impulse $F(\tau) d\tau$,

$$Mdv/dt = F(\tau) d\tau.$$

This means that due to the impulse during time change $d\tau$, the velocity changes by $F(\tau) d\tau/M$. This describes initial conditions that at $t = \tau$, $v = 0$, and $dv/dt = F(\tau) d\tau/M$.

Through this, we can utilise the previous solution for the case of damped free vibration given by (9). But since (9) is for $t = 0$, we must replace t in (9) with $t - \tau$. Also,

$$\begin{aligned} dv/dt &= [F(\tau)/M] d\tau, \text{ but from (17) we get} \\ &= -(d^2v_g/d^2\tau) d\tau. \end{aligned}$$

Making these substitutions in (9) we get,

$$v(t) = [-(d^2v_g/d^2\tau) d\tau/\omega_D] \exp [(-\xi\omega(t - \tau) \sin \omega_D(t - \tau))] \quad (18)$$

However, (18) is the solution to the system under the ground motion as an impulsive load. Hence when $F(t)$ is applied continuously, the solution is obtained by integrating (18). Therefore,

$$v(t) = - (1/\omega_D) \int_0^t [-(d^2v_g/d^2\tau)] \exp [(-\xi\omega(t - \tau) \sin \omega_D(t - \tau))] d\tau \quad (19)$$

This equation is called the Duhamel integral. It is in the class of the convolution integrals.

Since $\xi \ll 1$ in most building structures, $(1 - \xi^2)^{1/2} \approx 1$, so the undamped frequency can replace the damped frequency,

$$v(t) \approx - (1/\omega) \int_0^t [-(d^2v_g/d^2\tau)] \exp [(-\xi\omega(t - \tau) \sin \omega(t - \tau))] d\tau \quad (20)$$

The advantage of this is that it enables a simple relationship between the displacement, velocity and acceleration, with great practical application, as will be seen shortly.

As for the displacement, the velocity is approximately given as,

$$dv(t)/dt \approx - (1/\omega) \int_0^t [-(d^2v_g/d^2\tau)] \exp [(-\xi\omega(t - \tau) \cos [\omega(t - \tau) + \Psi]] d\tau \quad (21)$$

where $\Psi = \tan^{-1} \xi/(1 - \xi^2)^{1/2}$

Also, the absolute acceleration is given, when neglecting the second term on the right side of (14), by

$$d^2v/dt^2 + d^2v_g/dt^2 \approx \omega \int_0^t [(d^2v_g/d^2\tau)] \exp [(-\xi\omega(t - \tau) \sin \omega(t - \tau))] d\tau \quad (22)$$

3.3 Response Spectra

The aforementioned equations are of interest for the design of earthquake resistant structures by performance-based approaches but not for the design of such structures by typical design office methods. Design office methods are based on Statics, so a prime interest is the ability to use these equations under conditions that can be considered to be equivalent to Statics. Therefore, this interest becomes the search for the maximum response quantities, rather than the entire response from $t=0$. Through (20) to (22) this is possible.

The relative displacement v reaches its maximum when in (20), the integral takes the maximum value. With the maximum value of this quantity defined as S_v we have,

$$S_d = (1/\omega) S_v = v_{\max} \quad (23)$$

S_d is called the *spectral displacement*.

$$S_v = \max \text{ value of } \int_0^t [-(d^2v_g/d^2\tau)] \exp [(-\xi\omega(t - \tau) \sin \omega(t - \tau))] d\tau \quad (24)$$

In structures with damping S_v is not identical to the maximum velocity response, but is very close to it. Therefore S_v is considered equal to the maximum velocity and is called the *spectral pseudo velocity*.

According to (24) and (22),

$$S_a = \omega S_v \quad (25)$$

S_a is called the *spectral pseudo acceleration*.

Hence from (23),

$$S_a = \omega^2 S_d \quad (26)$$

The earthquake load applied to the structure, the maximum base shear V_{\max} ,

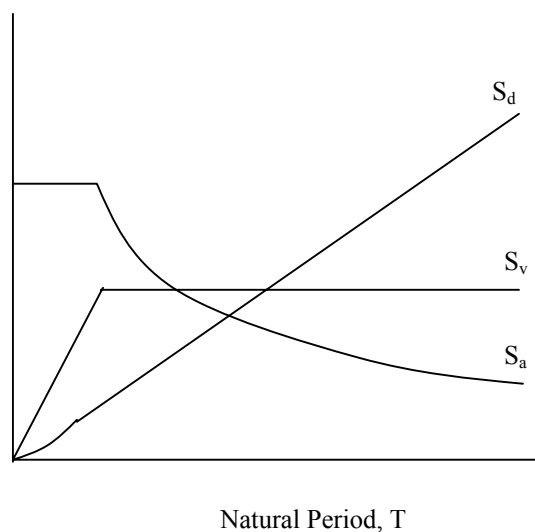
$$V_{\max} = MS_a \quad (27)$$

Therefore by knowing S_d the design load is easily known.

By using (23) to (26), S_a , S_d , and S_v for an SDOF system subjected to an earthquake motion can be drawn with respect to each particular combination of natural period and damping coefficient. This gives, in one single graph, the maximum response that the structure will experience. Such a graph is called a *response spectrum* and so the graphs of S_a , S_d , and S_v versus T are called the acceleration response spectrum, the displacement response spectrum, and the velocity response spectrum, respectively.

For convenience, these spectra are sometimes put on one logarithmic graph with 3 y-axes rotated relative to each other. Such a combined graph is called a *tripartite plot*. An example is shown in the Appendix.

As a single ground acceleration graph is necessarily jagged, a response spectrum graph for a single earthquake is also jagged. By considering many ground acceleration records, many spectra are assembled and statistical techniques applied to smooth out the jagged lines. This is very useful for preparing spectra for use in design and seismic design codes specify such spectra and rules for their construction. Also, certain general characteristics become apparent by the smoothing. These general characteristics of the spectra are shown below.



4.0 The Horizontal Vibration of a MDOF Structure

It is often more appropriate to model a structure as a set of lumped-masses rather than as a single lumped mass. This is particularly important for structures with significant variation of mass or stiffness with height.

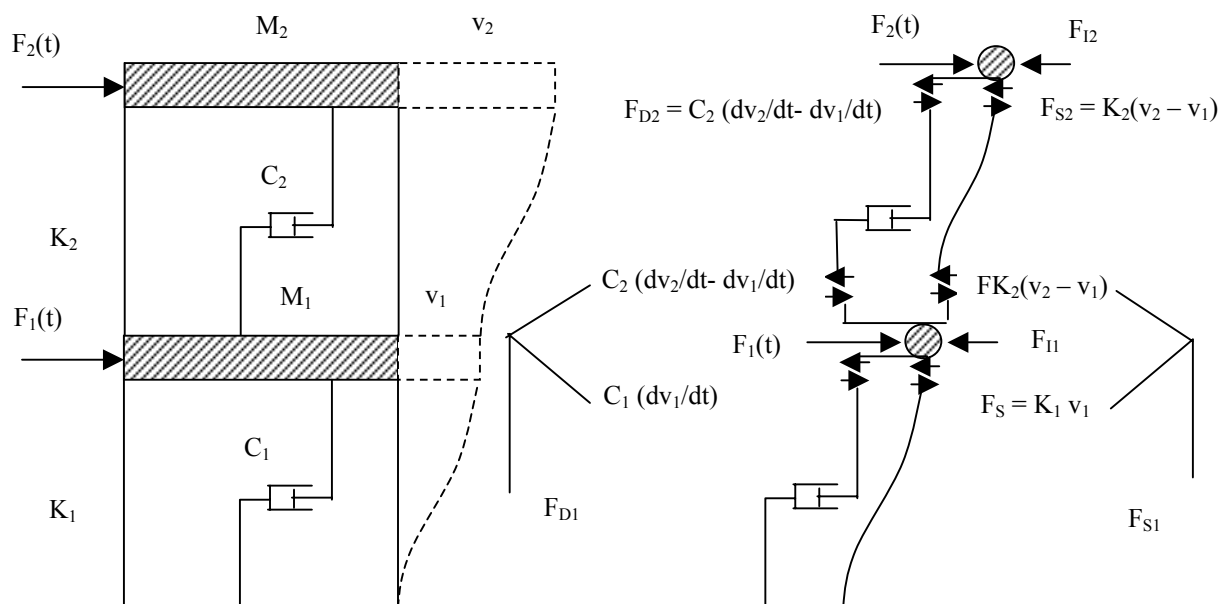
To maximise the economy of processing the calculations, it is desirable to solve the MDOF vibration problem by utilising the solutions of the SDOF problem. This is readily accomplished in the *Modal Analysis Technique* which is based on the following steps:

1. Determine the modes of vibration of the system.
2. Represent the displacement of the masses as the sum of the modal frequency times a Y vector, called the *general co-ordinate vector*, or the *normal coordinates*.
3. Make the system of equations independent of each other so that the system becomes a set of independent SDOFs. This is done by taking advantage of the fact that the mode shapes are *orthogonal*.
4. Use the previously presented solutions to SDOF problems to determine the response for each mode. This can give a *time history* response for each mode. Alternatively, use response spectra to determine the maximum responses. Since the modes of vibration for each mass do not peak at the same time, use a statistical technique to estimate the peak at each dof (each mass). The *square-root-sum-of-squares (SRSS)* formula is typically used for this (if the modal frequencies are not close).

It must be pointed out that since step 2 is essentially superposition, modal analysis is only valid for linear systems.

Also in step 2, when we apply the normal coordinates we must also do so for the ground acceleration term on the right-hand side. This results in a distribution of the ground acceleration to the various mode shapes in proportion to the *participation* of each mode shape to the total vibration.

In the following, we implement these steps but focus on the salient results of the steps. The student is referred to standard texts for the detailed derivations.



4.1 Modal Time-History Analysis

From the free body diagram, the governing equations of motion for a MDOF system can be represented as:

$$[M]d^2[v]/dt^2 + [C][v] + [K][v] = -[M][1]d^2v_g/dt^2 \quad (1)$$

This is analogous to the SDOF equation (2) but with the mass, damping and stiffness constants now replaced by matrices. $[1]$ is the unit diagonal vector.

Equation (1) can be uncoupled if $[v]$ is transformed using an orthogonal vector (refer to the end of this section for notes on orthogonality). It can be shown that such a vector corresponds to the displacements of an undamped system under free vibration.

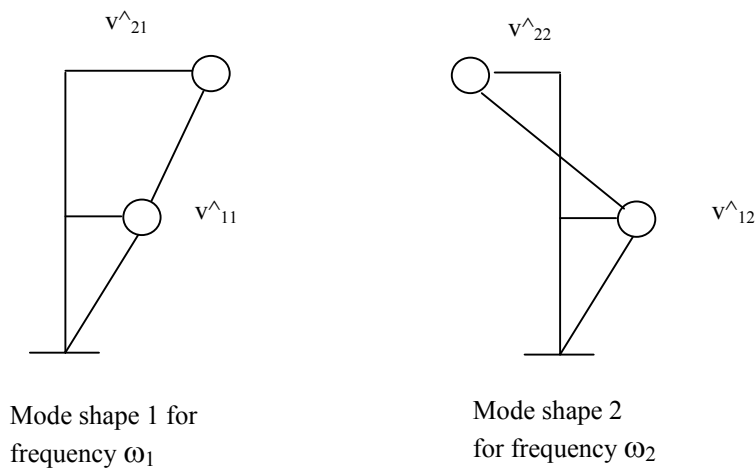
For an undamped system under free vibration (1) is solved by

$$[v] = [v^{\wedge}] \sin \omega t \quad (2)$$

This results in,

$$[K][v^{\wedge}] - \omega^2 [M][v^{\wedge}] = [0] \quad (3)$$

(3) is called the *frequency equation* and is an eigenvalue system with respect to the circular frequency ω . When the system has n degrees of freedom, n natural frequencies are obtained from (3). Each frequency results in a different set of displacements v^{\wedge} and the displacement profile for that frequency is called a *mode shape*. For a 2-DOF system we get,



Solution of (3) is typically carried out by computer except for the simplest of cases. From (3), the v^{\wedge} cannot be uniquely determined for each ω .

Therefore since it is the relative values that are important, it is customary to normalize or scale $[v^{\wedge}]$ in such a manner that the displacement corresponding to the top story is taken as unity. If the system has N dof, the n^{th} mode shape (remembering that each mode shape has N dof), is written as $[\phi_n]$ defined as,

$$[\phi_n] = \begin{bmatrix} \phi_{1n} \\ \phi_{2n} \\ \vdots \\ \phi_{Nn} \end{bmatrix} \equiv$$

$$(1/v^k_n) \begin{bmatrix} v^1_n \\ v^2_n \\ \vdots \\ v^N_n \end{bmatrix}$$

The assembly of all the ϕ_n results in a square matrix of all the n mode shape vectors. This matrix is called the *modal shape matrix*.

$$[\phi] = [\phi_1 \ \phi_2 \ \dots \ \phi_N] = \begin{bmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N1} & \phi_{N2} & \dots & \phi_{NN} \end{bmatrix}$$

Returning to the solution of (1), the objective is to uncouple these simultaneous equations, or make them independent of each other. If two vectors are independent of each other, the coefficient matrix in their matrix representation is a diagonal matrix. As an arbitrary example, if we have two simultaneous equations,

$$3x + 0y = 4 \quad \text{and}$$

$0x + 7y = 5$, then clearly $x = 4/3$ and $y = 5/7$. As the equations are separate or uncoupled the solutions are simple as we have sets of one variable with one unknown. Putting this in matrix form,

$$\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \text{or,} \quad [A][Y] = [B] \quad (4)$$

Hence the coefficient matrix is seen to be a diagonal matrix. Therefore to use this approach of uncoupling (1) we must introduce a new vector referred to as the Y vector or normal vector (represented by the x,y or [Y] in the above equation). So we transform [v] in equation (1) as,

$$[v] = \sum^N [\phi_n] Y_n \quad (5)$$

$$\text{or } [\phi][Y] = [v] \quad (6)$$

Now to make this procedure work, we must make (6) take a form like (4). For a matrix [A], if [φ] is orthogonal, then the operation $[\phi]^T [A] [\phi]$ will diagonalise that [A] matrix (from the Diagonalisation Theorem). Hence pre-multiplying (6) by $[\phi]^T [M]$ we get,

$$[\phi]^T [M] [\phi] Y_n = [\phi]^T [M] [v] \quad (7)$$

Hence $[\phi]^T [M] [\phi]$ is a diagonal matrix, and the Y_n are made independent or uncoupled.

$$Y_n = ([\phi]^T [M] / [\phi]^T [M] [\phi]) [v]$$

$$L_n = [\phi]^T [M] = \sum^N M_i \phi_i \quad (8)$$

$$M_n^* = [\phi]^T [M] [\phi] = \sum^N M_i \phi_i^2 \quad (9)$$

$$\alpha_n = L_n / M_n^* \quad (10)$$

α_n is called the *participation factor* since it represents the contribution of the n^{th} mode shape to the entire vibration. It should be mentioned however, that its value depends on the manner by which the modal displacements are normalized. M_n^* is called the *generalized mass*.

$M_{en} = L_n^2/M_n^*$ is called the *effective mass* for the n^{th} mode, and the sum of the effective masses for all the modes is equal to the total mass of the structure.

To solve (1) in normal coordinates, we must also express the right hand side in normal coordinates. Given the definitions of L_n and M_n^* ,

$$[1] = \sum^N [\phi_n] \alpha_n \text{ or } [1] = [\phi] [\alpha] \quad (11)$$

Substituting (6) and (11) into (1), and manipulating as for the SDOF case we get,

$$d^2 Y_n/dt^2 + 2\xi_n \omega_n dY_n/dt + \omega_n^2 Y_n = -\alpha_n d^2 v_g/dt^2 \quad (12)$$

By setting,

$$Y_n = \alpha_n Y_{no} \quad (13)$$

(12) becomes,

$$d^2 Y_{no}/dt^2 + 2\xi_n \omega_n dY_{no}/dt + \omega_n^2 Y_{no} = -d^2 v_g/dt^2 \quad (14)$$

This equation represents the culmination of the process of separating the equations of the MDOF problem into a set of n independent SDOF problems, and combining the results. According to (5) the displacements can be expressed as,

$$[v] = \sum^N [\phi_n] \alpha_n Y_{no} \quad (15)$$

Hence it can be shown that the velocities are approximately,

$$d[v]/dt = \sum^N [\phi_n] \alpha_n dY_{no}/dt \quad (16)$$

and the absolute accelerations are approximately,

$$d^2[v]/dt^2 + [1] d^2 v_g/dt^2 = -\sum^N [\phi_n] \alpha_n \omega_n^2 Y_{no} \quad (17)$$

Therefore for time-history ground acceleration analysis of a MDOF problem, the solution is obtained via (15) to (17), and SDOF time-history solutions (practically obtained from numerical integration).

NOTE ON ORTHOGONALITY:-

"Orthogonal" means "at right angles". We said earlier that if $[\phi]$ is an orthogonal matrix, then $[\phi]^T [\phi]$ is a diagonal matrix. In fact, this product equals the identity matrix which is a diagonal matrix with all diagonal terms equal to unity and off-diagonal terms equal to zero. If, the matrix $[\phi]$ is composed of vectors such that,

$$[\phi] = [\mathbf{c1} \ \mathbf{c2}]$$

$$[\phi]^T = \begin{bmatrix} \mathbf{c1} \\ \mathbf{c2} \end{bmatrix},$$

then noting the dot operator (\cdot) of vector multiplication (i.e. $\mathbf{a} \cdot \mathbf{b} = ab \sin\theta$)

$$[\phi]^T [\phi] = \begin{bmatrix} \mathbf{c1} \cdot \mathbf{c1} & \mathbf{c1} \cdot \mathbf{c2} \\ \mathbf{c2} \cdot \mathbf{c1} & \mathbf{c2} \cdot \mathbf{c2} \end{bmatrix} \text{ which is the unit matrix if } \mathbf{c1} \text{ and } \mathbf{c2} \text{ are unit vectors at right angles to each other. That is,}$$

$$[\phi]^T [\phi] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4.2 Modal Response Spectrum Analysis

If only maximum values are required, response spectra can be utilised as for the SDOF problem. Then for each mode n ,

$$S_{dn} = S_{vn}/\omega_n \quad (18)$$

$$v_n = \phi_n \alpha_n S_{dn} \quad (19)$$

$$d^2v_n/dt^2 = \omega_n^2 v_n \quad (20)$$

$$q_n = M_n d^2v_n/dt^2 \quad (21)$$

S_{dn} and S_{vn} are the spectral displacement and velocity respectively, and (19) to (21) are the n^{th} mode displacement, acceleration and inertia force respectively. For each dof (each mass), these quantities are then combined by statistical techniques with the most common being the SRSS combination expressed as, SRSS of the quantity $= \sqrt{(\sum^N \text{quantity}^2)}$. This is required since the modes of vibration for each mass do not peak at the same time.

4.2.1 Practical Application of Modal Response Spectrum Analysis to Earthquake Building Analysis

Dynamic analysis of a building is required if the building is irregular. A quantitative means of determining if a building is irregular is as follows:

1. Calculate the lateral inertia forces on the building at each floor as per the code ELF formula for vertical distribution of the base shear, V . The formula uses the height of each floor, h_i .
2. Calculate the storey shears.
3. Compute lateral displacements x_i at each floor under the forces from 1.
4. Substitute the x_i from 3 in place of the h_i in 1.
5. Re-calculate the story shears.
6. Compare the storey shears from 2 with those from 5 for each floor.
7. If for any floor the difference is more than 30%, then a dynamic analysis must be done.

If it is determined that a dynamic analysis is required, the most widely used approach for design purposes, is the modal response spectrum analysis method presented previously. (Equation numbers refer to the equations of section 4.1).

The procedure for its use is:

1. Model the structure as a MDOF shear building and determine the $[K]$ and $[M]$ matrices.
2. By using an eigenvalue solver or otherwise, obtain the modal circular frequencies.
3. By substitution in (3), determine the modal shape for each vibration mode and assemble the modal shape matrix.
4. Convert the modal frequencies from circular (radians per second) to rectilinear (Hertz) by dividing by 2π
5. For each mode shape, and from the response spectrum, determine S_{vn} or S_{dn} . This may require conversion of the result from 4, to the modal period.
6. For each mode shape determine: L_n (eq 8); the generalised mass M_n^* (9), and the participation factor α_n (10).
7. For each mode shape, and for each floor per mode shape, determine the displacement (eq (19)), the acceleration (20), and the inertia force (21).
8. For each floor combine the results of 7 for each mode shape by the SRSS combination.