CVNG 3001 STRUCTURAL ENGINEERING

STRUCTURAL DYNAMICS

OUTLINE OF TOPICS
(4 No. Sessions at 1 hr each)
by R. Clarke

Delivery Media: ☑ Oral ☑ Blackboard ☑ Handouts ☑ Transparencies ☑ Internet

Equipment: ☑ Transparency Projector ☐ Computer Projector

Outcome: The student shall appreciate vibratory effects on structures with an emphasis on earthquake application.

Scope/Limitations: Buildings; Base Excitation; Lumped-Mass; Closed-form Solutions; Linearity.

TOPICS

1.0 Fundamental Characteristics of Vibration
- 1.1 Free and Forced Vibration
- 1.2 Period and Frequency
- 1.3 Damping

2.0 Uses of Structural Dynamics
- 2.1 Structural Engineering Uses
- 2.2 Civil Engineering Uses
- 2.3 Earthquake Engineering Uses

3.0 The Horizontal Vibration of a SDOF Structure
- 3.1 Free Vibration – Transient Ground Acceleration
- 3.2 Forced Vibration – Steady-State Ground Acceleration
- 3.3 Response Spectra

4.0 The Horizontal Vibration of a MDOF Structure
- 4.1 Modal Time History Analysis
- 4.2 Modal Response Spectrum Analysis (RSA)
  - 4.2.1 Procedure for Earthquake Building Analysis by Modal RSA
  - 4.2.2 Example

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1. Fundamental Characteristics of Vibration

Vibration is a time-varying response to a change in motion of a structure in such a manner that inertia forces are imposed which resist the change. Only when inertia forces arise and affect the system, is the phenomenon considered a phenomenon of dynamics.

In the presence of inertia forces, the system is governed by Newton's Second Law such that it is in a state of dynamic equilibrium (d'Alembert's Principle). If the system is disturbed and the disturbance removed, the system tries to return to its former state and vibrates on its own until it does so. This "vibration on its own" is called *free vibration*. Vibration due to an external time-varying disturbance is called *forced vibration.* During free vibration, the structure moves from side to side and the time taken to move from one position, back to that original position, is called the *natural period.* Moving from one position, back to that original position, is called making a *cycle,* so the period, is the time required to make one cycle. Also, the number of cycles made in one second (Hertz, Hz), is the *frequency.*

The nature of dynamic equilibrium is that the system or structure always vibrates as a combination of basic deflected shapes of the structure. These basic shapes are called the *modes of vibration.* To see the modes of vibration of, for example, a guitar string, you pluck the string, then shine a strobe light on the string in such a way that the frequency of the light is the same as the frequency of the particular mode of vibration. The string appears to be still but bent, though in fact it is still vibrating – a *stationary wave*. Change the frequency of the strobe light to match that of the next mode of vibration of the string, and another shape appears. This can go on ad infinitum. The mode shape with the lowest frequency is called the *fundamental mode of vibration,* and that lowest frequency is likewise called the *fundamental frequency*. It is important to note that the mode shape is a stationary wave. The reason it is stationary is because when a system is disturbed it always tries to return to its undisturbed state. This tendency is present in any kind of vibration problem and this is why in solving structural vibration problems, the solution is always found as a combination of the modes of vibration.

This tendency manifests mathematically, in the solution of the dynamic equilibrium equations that govern the motion, as the solution to the well-known eigenvalue problem. Hence the modes of vibration are eigenvectors, and the modal frequencies are the corresponding eigenvalues. Therefore solving the attendant eigen-problem is a sub-set of any total solution of a vibration problem (except for the simplest of cases).

To complete the review of basic vibration phenomena, it must be mentioned that free vibration is always accompanied by *damping*. This is the decay in the free vibration over time due to friction. If there is too much damping, the structure will not oscillate and is called *over-damped.* The amount of damping that just causes over-damping is called the *critical damping*.

2. Uses of Structural Dynamics

Building and civil engineering facilities are comprised of structures. The main areas in which structural dynamics is applied are - earthquake engineering of building structural systems; earthquake engineering of building non-structural systems (e.g. machinery; suspended pipes, etc.); earthquake engineering of soils; earthquake engineering of infra-structural elements (e.g. bridges; culverts; tanks; etc.); earthquake engineering of lifeline systems (e.g. embedded pipes, etc.); wind engineering of buildings, transmission towers, etc.; blast engineering of structures, and coastal engineering in terms of the effects of waves on shore line and off-shore facilities.

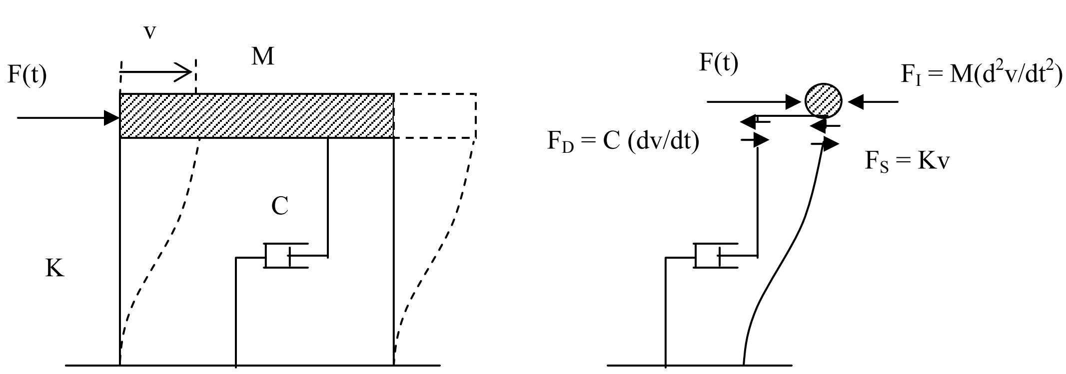
This presentation focuses on the area of earthquake engineering of building structural systems although the fundamental approaches and solutions apply to many of the other areas as well.

In this area, the following dynamic analysis approaches are used, in descending order of regularity. For this course, only the main aspects of modal analysis, and the MDOF Response Spectrum approaches are presented.

APPROACH	DESCRIPTION	LIMITATION	COMMENTS
1. SDOF Response Spectrum - ELF Procedure	The basic approach used in design codes; no time variation	For regular structures only; requires the use of a correlation variable, R_w, for typical earthquake design.	This is a special case of the MDOF Response Spectrum approach
2. MDOF Response Spectrum	A generalised version of the above.	For irregular structures; requires the use of a correlation variable, R_w	Derived from modal analysis but by focusing on peak values; very useful for a wide range of building design problems
3. Modal Analysis Time History	Uses the property of orthogonality to separate the motion into a set of independent motions	Linear. Analytical solutions only available for simple forcing functions but not earthquake excitation.	Provides results for any time, t, during the disturbance.
4. Direct Integration - Linear Elastic	The governing equations involve linear constitutive relations and are solved by numerical integration	Linear, but can solve for more complex forcing functions	Can be used with modal analysis. Also provides results for any time, t, during the disturbance.
5. Direct Integration - Non-linear	The governing equations are non-linear since they require non-linear constitutive relations; solved by numerical integration	More computer intensive since it requires an optimisation solver	Most accurate of the deterministic approaches; rapidly gaining prominence for performance-based design due to improved computer power/cost.
6. Probabilistic	The governing equations are probabilistic	The reliability estimates are not absolute but relative hence can only be used for comparing designs	Can be linear or non-linear. Used by researchers and code developers.

3.0 The Horizontal Vibration of a SDOF Structure

A SDOF system is simply a representation of the structure as a single lumped-mass that can only move one way. Hence it is called a single degree of freedom (SDOF) system. A building modelled in this way where the mass moves sideways is called a shear frame. This system embodies most of the concepts associated with the engineering mathematics of vibration. It is also a necessary introduction to modal analysis since modal analysis works by separating the structure into a set of separate SDOF's.



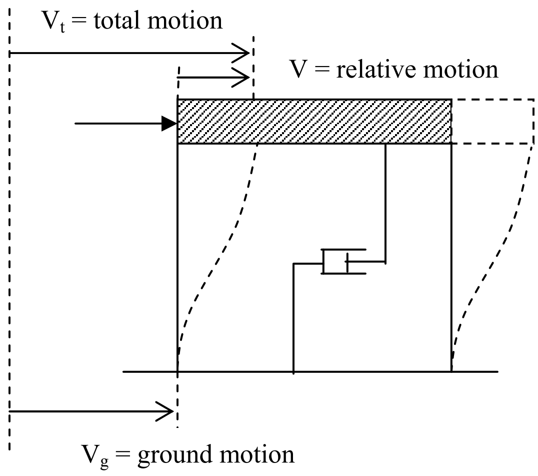
According to d'Alembert's principle,

$F_I + F_D + F_S = F(t)$

Where F_I = inertial force
F_D = damping force
F_S = force resisted by spring

Hence,

$M(d^2v/dt^2) + C\,(dv/dt) + Kv = F(t)$ (1)



For ground acceleration, the inertia force on the lumped mass is with respect to the total motion such that,

$F_I = M(d^2v/dt^2 + d^2v_g/dt^2)$

Therefore with respect to the forces on the lumped mass and hence in terms of the relative motion of the structure, eq(1) becomes,

$M(d^2v/dt^2 + d^2v_g/dt^2) + C\,(dv/dt) + Kv = 0$, or

$M(d^2v/dt^2) + C\,(dv/dt) + Kv = -Md^2v_g/dt^2$ (2)

Eq (2) is the governing equation of a structure under ground acceleration modeled as a SDOF shear building.

It is one of the most fundamental governing equations of structural dynamics. Now v and v_g are functions of time. Therefore, the various kinds of structural SDOF dynamics problems encountered are simply due to the definition of v_g as it varies with time:

When,

$v_g = 0$, we get the free body problem
v_g = a simple harmonic function such as $v_0 \sin \omega t$, we get the classic equations of damped vibration which mathematically displays the central features of resonance, over-damping etc.
v_g = any general function, we get the general solution of any damped vibration problem. This solution is called the Duhamel or convolution integral. The solution of these cases follow.

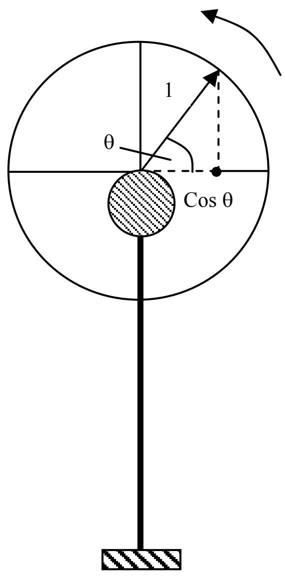
3.1 Free Vibration - Transient Ground Acceleration

The free vibration problem is also called the transient response problem. This is because just after t=0, when the ground acceleration begins, regardless of the type of ground acceleration, this initial value is like an impulse acceleration which the system responds to as a free vibration superimposed on the subsequent ground acceleration. The effect gradually fades so is called the transient response. After fading away, the response is called the steady-state response.

If no ground motion is applied to an SDOF system without damping, eq(2) is simplified to,

$M(d^2v/dt^2) + Kv = 0$ (3)

Simpler solutions tend to be obtained if trigonometric functions are used to solve (3) for the displacement v. Since the displacement of the mass occurs along a straight line, the unit circle concept shows how this is achieved.



Consider the unit circle with the radius rotating anti-clockwise ω times per second. The displacement is the horizontal projection of the unit radius. The projection follows the radius as it rotates such that when the radius is at 0 deg, the projection length is the same as the radius (=1), and when the radius is at 90 deg, the projection length is zero. In other words, as the unit radius rotates, the end of the projection length (the dot) vibrates.

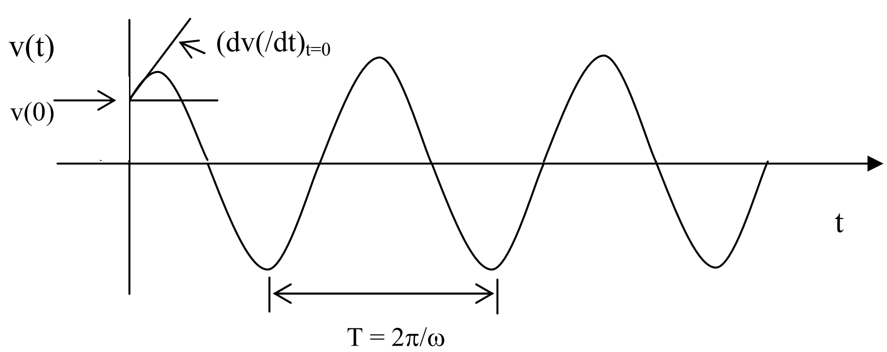
Since the unit radius makes ω revolutions per second this implies that the radius rotates ωt times in t seconds. But for one revolution θ is 2π and the time taken is the period, T. Hence,

$\omega T = 2\pi$

ω is therefore called the *natural circular frequency* and is of unit radians per second.

Substituting the harmonic relation, $v = A \cos \omega t + B \sin \omega t$, shows that this is the general solution to (3). From the trig identity $A\cos\theta - B\sin\theta = C\cos(\theta + \alpha)$, it can be shown that the peak value or amplitude of v is $(A^2+B^2)^{1/2}$. Also, $\omega = \sqrt{(K/M)}$, hence $T = 2\pi/\omega = 2\pi (M/K)^{1/2}$.

The constants A and B are determined from the initial conditions. At t = 0, the constants A and B are v(0), $[(dv/dt)_{t=0}]/\omega$, respectively.



If viscous damping is present in the system, (3) becomes,

$M(d^2v/dt^2) + C\,(dv/dt) + Kv = 0$ (4)

Dividing by M, we get,

$d^2v/dt^2 + 2\xi\omega(dv/dt) + \omega^2 v = 0$ (5)

where $2\xi\omega = C/M$

The solution of (5) is,

$v = A \exp(\lambda_1 t) + B \exp(\lambda_2 t)$

where $\lambda_1, \lambda_2 = \omega[-\xi \pm (\xi^2 - 1)^{1/2}]$ (6)

(6) indicates that the solution changes form according to the value of ξ.

If $\xi^2 < 1$,

$v = C \exp(-\xi\omega t) \sin(\omega_D t + \theta)$ (7)

where $C = (A^2 + B^2)^{1/2}$ $\theta = \tan^{-1} A/B$

$\omega_D = (1 - \xi^2)^{1/2} \omega$ (8)

For initial conditions of v = 0 and $(dv/dt)_{t=0}$ at t=0, (7) becomes

$v = [(dv/dt)_{t=0}/\omega_D] \exp(-\xi\omega t) \sin \omega_D t$ (9)

ω_D is called that the *damped circular frequency*. The portion of the equation before the sine of (7) and (9), indicates that the system experiences a decaying oscillation with t.

If $\xi^2 > 1$, the system does not oscillate since the effect of the damping overshadows the oscillation (over-damping).

The condition $\xi^2 = 1$ indicates a limiting value of damping at which the system loses its vibratory characteristics; this is called *critical damping*. From (5), the critical damping constant,

$C_{cr} = 2\omega M = 2(MK)^{1/2}$ (10)

ξ is defined in terms of C_{cr} as, $\xi = C/C_{cr}$. Hence ξ is called the *fraction of critical damping,* or the *damping ratio*. Typical values of ξ are 5% for reinforced concrete, and 2% for steel.

From (7) we get a means for experimentally determining ξ. It can be shown that for successive vibration amplitudes of a damped system,

$\ln(v_n/v_{n+1}) \approx 2\pi\xi$. This is called the *logarithmic decrement.*

3.2 Forced Vibration - Steady-State Ground Acceleration

If the system is subjected to a sinusoidal ground motion, then after the aforementioned transient response has dissipated, the steady-state response is obtained.

$v_g = \alpha_0 \sin \omega' t$, where ω' is the frequency of the forcing function,

Hence,

$d^2v/dt^2 + 2\xi\omega(dv/dt) + \omega^2 v = - \alpha_0 \sin \omega' t$ (11)

The solution of (11) is the sum of the complimentary solution (7), and the particular solution. The particular solution is,

$v = C_1 \sin \omega' t + C_2 \cos \omega' t$ (12)

Substituting (12) into (11),

$v = - (\alpha_0/\omega^2) [(1 - \beta^2)^2 + 4\xi^2\beta^2]^{-1/2} \sin(\omega' t - \theta)$ (13)

where $\theta = \tan^{-1} (2\xi\beta/(1 - \beta^2))$

$\beta = \omega'/\omega$

It is instructive to investigate the effect of β on the increase in the load, called the *amplification*, due to the influence of the inertia.

A static external force equal to the inertial force ($M\alpha_0$) makes the system deform by $M\alpha_0/K = \alpha_0/\omega^2$. This deflection, labeled v_{st} is,

$$v_{st} = \alpha_0/\omega^2$$

Hence according to (13) the ratio of the dynamic to static deflection, called the *dynamic displacement amplification factor,* is

$$D_d = |v/v_{st}| = [\ (1 - \beta^2)^2 + 4\xi^2\beta^2]^{-1/2}$$

When D_d is plotted as a function of the frequency ratio β, and for different ξ, we get Appendix Fig 1.

Note that as ω' approaches ω the amplitude increases; this tendency is greater for smaller values of ξ. The condition at which $\omega' = \omega$, or $\beta = 1$, is called *resonance*.
It was stated earlier that the absolute acceleration is $d^2v/dt^2 + d^2v_g/dt^2$, hence from (2),

$$d^2v/dt^2 + d^2v_g/dt^2 = -\ (\ \omega^2\ v + 2\xi\omega\ (dv/dt)\ ) \qquad (14)$$

Substituting (13) and its first derivative into (14) we get

$$d^2v/dt^2 + d^2v_g/dt^2 = \alpha_0[\ (1 - \beta^2)^2 + 4\xi^2\beta^2]^{-1/2}\ [1 + 4\xi^2\beta^2]^{1/2}\ \sin[\ \omega'\ t - (\theta - \theta_0)] \qquad (15)$$

where $\theta_0 = \tan^{-1}\ 2\xi\beta$. Thus the ratio of the response acceleration to the ground acceleration is,

$$D_a = |\ (d^2v/dt^2 + d^2v_g/dt^2)/d^2v_g/dt^2| = [\ (1 - \beta^2)^2 + 4\xi^2\beta^2]^{-1/2}\ [1 + 4\xi^2\beta^2]^{1/2} \qquad (16)$$

D_a is called the *dynamic acceleration-magnification factor* and when D_a is plotted as a function of the frequency ratio β, and for different ξ, we get Appendix Fig 2. At $\beta = \sqrt{2}$ D_a is unity regardless of the values of ξ, and D_a becomes smaller than unity in the range $\beta > \sqrt{2}$.

Forced Vibration - Non-Steady-State Ground Acceleration

For the case where the ground acceleration is an arbitrary general function, as is the case in real earthquakes, the solution is obtained by a technique in which the ground motion is considered to correspond to the sum of a series of impulsive loads. However, an impulsive load problem is the same as a free vibration problem but with different initial conditions.

The effective external force caused by arbitrary ground motion is,

$$F(t) = -Md^2v_g/dt^2 \qquad (17)$$

Taking F(t) as an impulsive load applied during an infinitesimal time interval $d\tau$, and from the condition that the momentum Mdv/dt equals the impulse $F(\tau)\ d\tau$,

$$Mdv/dt = F(\tau)\ d\tau.$$

This means that due to the impulse during time change $d\tau$, the velocity changes by $F(\tau)\ d\tau/M$. This describes initial conditions that at $t = \tau$, $v = 0$, and $dv/dt = F(\tau)\ d\tau/M$.

Through this, we can utilise the previous solution for the case of damped free vibration given by (9). But since (9) is for $t = 0$, we must replace t in (9) with $t - \tau$. Also,

$dv/dt = [F(\tau)/M]\ d\tau$, but from (17) we get

$$= -(d^2v_g/d^2\tau)\ d\tau.$$

Making these substitutions in (9) we get,

$$v(t) = [-(d^2v_g/d^2\tau)\ d\tau/\omega_D]\ \exp\ [(-\xi\omega(t - \tau)\ \sin\ \omega_D(t - \tau)] \qquad (18)$$

However, (18) is the solution to the system under the ground motion as an impulsive load. Hence when F(t) is applied continuously, the solution is obtained by integrating (18). Therefore,

$$v(t) = -\ (1/\omega_D) \int_0^t [-(d^2v_g/d^2\tau)\ ]\ \exp\ [(-\xi\omega(t - \tau)\ \sin\ \omega_D(t - \tau)]\ d\tau \qquad (19)$$

This equation is called the Duhamel integral. It is in the class of the convolution integrals.

Since $\xi << 1$ in most building structures, $(1 - \xi^2)^{1/2} \approx 1$, so the undamped frequency can replace the damped frequency,

$$v(t) \approx -\ (1/\omega) \int_0^t [-(d^2v_g/d^2\tau)\ ]\ \exp\ [(-\xi\omega(t - \tau)\ \sin\ \omega(t - \tau)]\ d\tau \qquad (20)$$

The advantage of this is that it enables a simple relationship between the displacement, velocity and acceleration, with great practical application, as will be seen shortly.

As for the displacement, the velocity is approximately given as,

$$dv(t)/dt \approx -\ (1/\omega) \int_0^t [-(d^2v_g/d^2\tau)\ ]\ \exp\ [(-\xi\omega(t - \tau)\ \cos\ [\omega(t - \tau) + \Psi]\ d\tau \qquad (21)$$

where $\Psi = \tan^{-1} \xi/(1 - \xi^2)^{1/2}$

Also, the absolute acceleration is given, when neglecting the second term on the right side of (14), by

$$d^2v/dt^2 + d^2v_g/dt^2 \approx \omega \int_0^t [(d^2v_g/d^2\tau)\ ]\ \exp\ [(-\xi\omega(t - \tau)\ \sin\ \omega(t - \tau)\ d\tau \qquad (22)$$

3.3 Response Spectra

The aforementioned equations are of interest for the design of earthquake resistant structures by performance-based approaches but not for the design of such structures by typical design office methods. Design office methods are based on Statics, so a prime interest is the ability to use these equations under conditions that can be considered to be equivalent to Statics. Therefore, this interest becomes the search for the maximum response quantities, rather than the entire response from t=0. Through (20) to (22) this is possible.

The relative displacement v reaches its maximum when in (20), the integral takes the maximum value. With the maximum value of this quantity defined as S_v we have,

$$S_d = (1/\omega)\ S_v = v_{max} \qquad (23)$$

S_d is called the *spectral displacement.*

$$S_v = \text{max value of} \int_0^t [-(d^2v_g/d^2\tau)\ ]\ \exp\ [(-\xi\omega(t - \tau)\ \sin\ \omega(t - \tau)]\ d\tau \qquad (24)$$

In structures with damping S_v is not identical to the maximum velocity response, but is very close to it. Therefore S_v is considered equal to the maximum velocity and is called the *spectral pseudo velocity.*

According to (24) and (22),

$$S_a = \omega S_v \tag{25}$$

S_a is called the *spectral pseudo acceleration.*

Hence from (23),

$$S_a = \omega^2 S_d \tag{26}$$

The earthquake load applied to the structure, the maximum base shear V_{max},

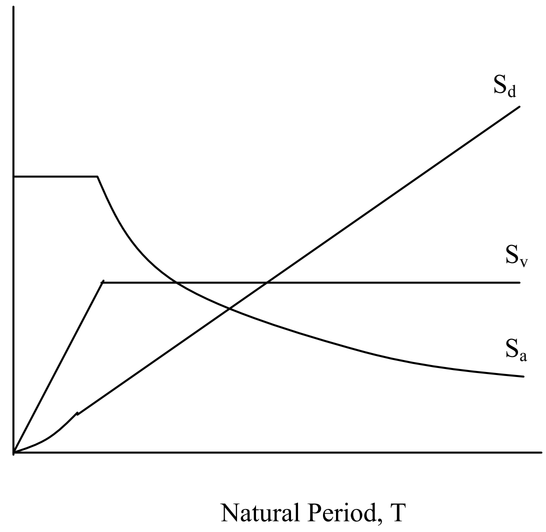
$$V_{max} = MS_a \tag{27}$$

Therefore by knowing S_d the design load is easily known.

By using (23) to (26), S_a, S_d, and S_v for an SDOF system subjected to an earthquake motion can be drawn with respect to each particular combination of natural period and damping coefficient. This gives, in one single graph, the maximum response that the structure will experience. Such a graph is called a *response spectrum* and so the graphs of S_a, S_d, and S_v versus T are called the acceleration response spectrum, the displacement response spectrum, and the velocity response spectrum, respectively.

For convenience, these spectra are sometimes put on one logarithmic graph with 3 y-axes rotated relative to each other. Such a combined graph is called a *tripartite plot.* An example is shown in the Appendix.

As a single ground acceleration graph is necessarily jagged, a response spectrum graph for a single earthquake is also jagged. By considering many ground acceleration records, many spectra are assembled and statistical techniques applied to smooth out the jagged lines. This is very useful for preparing spectra for use in design and seismic design codes specify such spectra and rules for their construction. Also, certain general characteristics become apparent by the smoothing. These general characteristics of the spectra are shown below.



4.0 HORIZONTAL VIBRATION OF A MDOF STRUCTURE - THE SHEAR BEAM MODEL

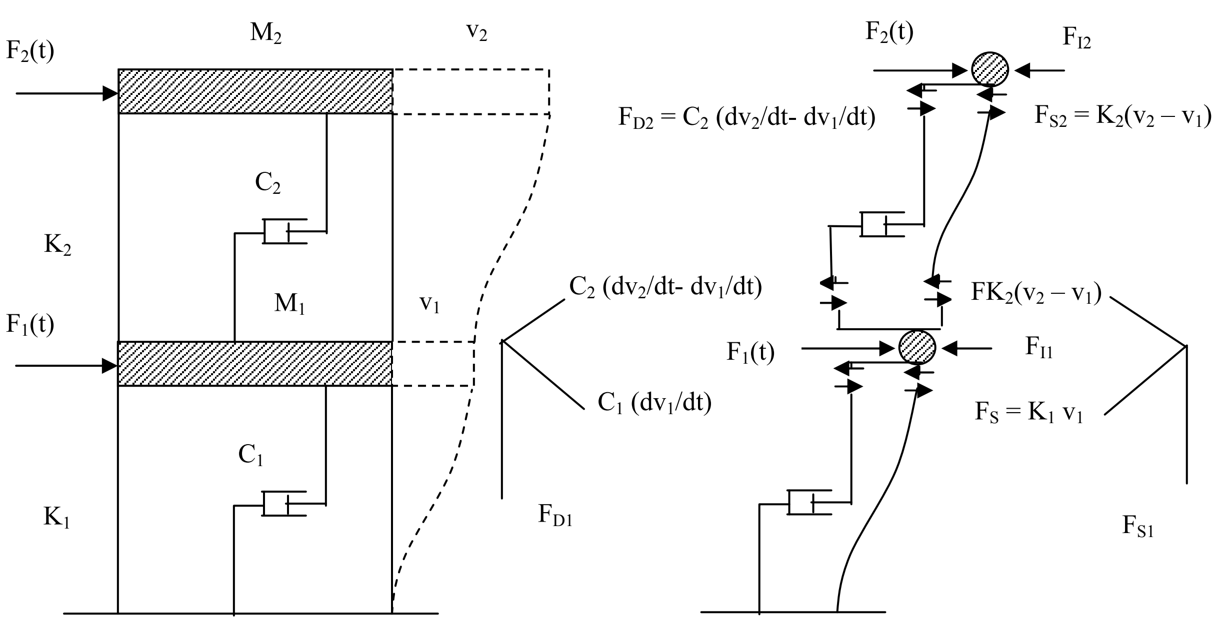
For earthquake-resistant design, building codes require that the magnitude and distribution of the earthquake forces on a structure that is vertically irregular in terms of mass or stiffness, be determined by dynamic analysis. If this is the only form of irregularity, then modeling the structure in terms of plane frames, walls (or dual systems) is appropriate. In such cases, it is commonplace to model the mass of the structure as a set of lumped-masses with one at each floor. Since the model is 2D there is then one DOF per floor which is translational only. Such a model of the structure is called a *shear beam* model.

To maximise the economy of processing the calculations, it is desirous to solve the MDOF vibration problem by utilising the solutions of the SDOF problem. This is readily accomplished using the *Modal Analysis Technique* which is based on the following steps for earthquake vibration:

1. Determine the modes of vibration of the system.
2. Represent the displacement of the masses as the sum of the modal displacement times a Y vector, called the *general co-ordinate vector,* or the *normal coordinates*.
3. Make the system of equations independent of each other so that the system becomes a set of independent SDOFs. This is done by taking advantage of the fact that the solution to the MDOF free vibration problem, i.e. the mode shapes, are *orthogonal*.
4. Use the solutions to SDOF problems to determine the response for each mode. This can give a *time history* response for each mode. Alternatively, use response spectra to determine the maximum responses. Since the modes of vibration for each mass do not peak at the same time, use a statistical technique to estimate the peak at each dof (each mass). The *square-root-sum-of-squares (SRSS)* formula is regularly used for this.

It must be pointed out that since step 2 is essentially superposition, modal analysis is only valid for linear systems.

In the following, we implement these steps but focus on the salient results of the steps. The student is referred to standard texts for the detailed derivations.



From the free body diagram, the governing equations of motion for a MDOF system can be represented as:

$$[M]d^2[v]/dt^2 + [C][v] + [K][v] = -[M][1]d^2v_g/dt^2 \qquad (1)$$

This is analogous to the SDOF equation but with the mass, damping and stiffness constants now replaced by (n x n) matrices. [1] is the (n x 1) unit vector and is required to ensure that the right-hand side of (1) is (n x 1).

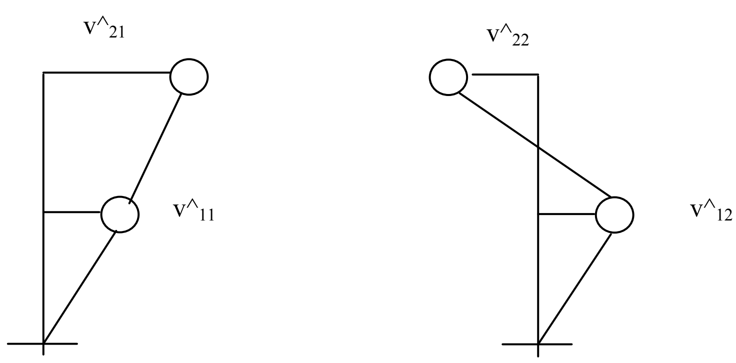
For an undamped system in free vibration, it can be shown that (1) is solved by

$$[v] = [v\hat{}]\sin \omega t \qquad (2)$$

This results in,

$$[K][v\hat{}] - \omega^2 [M]][v\hat{}] = [0] \qquad (3)$$

(3) is called the *frequency equation* and is an eigenvalue system with respect to the circular frequency ω. When the system has n degrees of freedom, n natural frequencies are obtained from (3). Each frequency results in a different mode shape. For a 2-DOF system we get,



Mode shape 1 for frequency ω_1

Mode shape 2 for frequency ω_2

Solution of (3) is typically carried out by computer except for the simplest of cases. From (3), it is the ratio of the displacements $v\hat{}_2/v\hat{}_1$ that are uniquely determined for each ω, and not the values themselves.

Therefore since it is the relative values that are important, it is customary to scale [v^] is such a manner that the displacement corresponding to the top story is taken as unity. If the system has N dof, the n^{th} modal shape (remembering that each modal shape has N dof), is written as $[\phi_n]$ defined as,

$$[\phi_n] = \begin{bmatrix} \phi_{1n} \\ \phi 2_n \\ . \\ . \\ \phi_{Nn} \end{bmatrix} \equiv$$

$$(1/v\hat{}_{kn}) \begin{bmatrix} v\hat{}_{1n} \\ v\hat{}_{2n} \\ . \\ . \\ v\hat{}_{Nn} \end{bmatrix}$$

Note that each ϕ_n is just the scaled displacement solution for the undamped free vibration of the structure. The assembly of all the ϕ_n results in a square matrix of all the n mode shape vectors. This matrix is called the *modal shape matrix*. Rows are dof's, and columns are modes.

$$[\phi] = [\phi_1\ \phi_2 \ldots\ \phi_N] = \begin{bmatrix} \phi_{11} & \phi_{12} & \ldots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \ldots & \phi_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ \phi_{21} & \phi_{22} & \ldots & \phi_{NN} \end{bmatrix}$$

The objective is to separate the modal responses, or make them independent of each other for solution. If two vectors are independent of each other, the coefficient matrix in their matrix representation is a diagonal matrix. As an arbitrary example, if we have two simultaneous equations,

$3x + 0y = 4$ and
$0x + 7y = 5$, then clearly $x = 4/3$ and $y=5/7$. As the equations are separate the solutions are simple as we have sets of one variable with one unknown. Putting this in matrix form,

$$\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix} \quad \text{or,} \quad [A]\,[Y] = [B] \qquad (4)$$

Hence the coefficient matrix is seen to be a diagonal matrix. Therefore to use this approach we must introduce a new vector referred to previously as the Y vector and represented by the x,y or [Y] in the above equation. So we transform [v] in equation (1) as,

$$[v] = \Sigma^N [\phi_n]\, Y_n \qquad (5)$$

$$\text{or } [\,\phi\,]\,[Y] = [v] \qquad (6)$$

It is from (6) that this dynamic analysis approach gets its name - Modal Superposition. As the [ϕ] matrix is (n x n) and the [Y]matrix is (n x 1), the [v] matrix is (n x 1). Because the Modal Analysis method uses superposition, it is only applicable to linear behavior. Now to make this procedure work, we must make (1) take a form like (4). This would happen if [ϕ] is orthogonal because for an orthogonal matrix [ϕ], pre and post-multiplying a matrix by $[\,\phi\,]^T$ and [ϕ] respectively, results in a diagonal matrix. Since [ϕ] satisfies the eigenvalue equation given by (3) above, it is orthogonal hence can be used to diagonalize [M], [C], and [K] on the left-hand side of (1). Therefore by substituting (6) in (1) and pre-multiplying by $[\,\phi\,]^T$ we get,

$$[\,\phi\,]^T [M]\,[\,\phi\,]\, d^2[Y]/dt^2 + [\,\phi\,]^T [C]\,[\,\phi\,]\, d[Y]/dt + [\,\phi\,]^T [K]\,[\,\phi\,]\,[Y] = -\,[\,\phi\,]^T [M]\,[1]\, a_g \ (7)$$

The diagonalized form of [M], [C] and [K] are therefore given by, $[M_{dia}] = [\,\phi\,]^T [M]\,[\,\phi\,]$, $[C_{dia}] = [\,\phi\,]^T [C]\,[\,\phi\,]$, and $[K_{dia}] = [\,\phi\,]^T [K]\,[\,\phi\,]$ resulting in,

$$[M_{dis}]\, d^2[Y]/dt^2 + [C_{dia}]\,[Y]/dt + [K_{dia}]\,[Y] = -\,[\,\phi\,]^T [M]\,[1]\, a_g \qquad (7b)$$

Let $[\alpha] = (\,[\,\phi\,]^T [M][1]\,) / (\,[\,\phi\,]^T [M]\,[\,\phi\,]\,)$, where $[\alpha]$ is an (n x 1) matrix. Therefore, (7b) becomes

$$[M_{dis}]\, d^2[Y]/dt^2 + [C_{dia}]\,[Y]/dt + [K_{dia}]\,[Y] = -\,[M_{dia}]\,[\alpha]\, a_g \qquad (7c)$$

Dividing by $[M_{dia}]$ and noting that the left-hand side of (7c) are independent equations,

$$d^2Y_n/dt^2 + 2\,\xi_n\,\omega_n\, dY_n/dt + \omega_n^2\, Y_n = -\,[\alpha]\, a_g = -\,(\Sigma^N M_i\,\phi_i)/(\,\Sigma^N M_i\,\phi_i^2)\, a_g \qquad (7d)$$

$$L_n = \Sigma^N M_i\,\phi_i \qquad (8)$$

$$M_n^* = n^{th} \text{ mode } \textit{Generalised Mass} = \Sigma^N M_i\,\phi_i^2 \qquad (9)$$

$$\alpha_n = \underline{n^{th} \text{ mode Participation factor}} = L_n/M_n^* \qquad (10)$$

n^{th} mode Generalised Mass:

"n" refers to the "n"th mode and "i" refers to the "i"th floor. α_n is called the n^{th} mode *participation factor*. Sometimes L_n is called the participation factor. This is the case when the scaling of the mode shape is done such that the denominator of (10) is unity, and the resulting mode shapes are called "mass normal". This approach is sometimes presented in the textbooks and is felt to make the calculations easier to perform. In the presentation above, the scaling was done simply by dividing by $v^\wedge_{kn}$.

$$M_{eff,n} = \alpha_n L_n = (\Sigma^N M_i \phi_i)^2 / (\Sigma^N M_i \phi_i^2) \quad (11)$$

$M_{eff,n}$ is called the *effective mass* of the structure involved in the n^{th} mode and is a parameter that is used in building codes to indicate the minimum number of modes that must be considered in the calculation. Recalling equation (7d),

$$d^2Y_n/dt^2 + 2\xi_n\omega_n\, dY_n/dt + \omega_n^2\, Y_n = -\alpha_n\, a_g \quad (12)$$

This equation is not yet identical to the SDOF equation of motion, but it is the intention of modal analysis to use the SDOF solutions to solve the MDOF problem.

By setting,

$$Y_n = \alpha_n\, Y_{no} \quad (13)$$

(12) becomes,

$$d^2Y_{no}/dt^2 + 2\xi_n\omega_n\, dY_{no}/dt + \omega_n^2\, Y_{no} = -a_g \quad (14)$$

4.1 Time History Analysis

This equation represents the culmination of the process of separating the equations of the MDOF problem into a set of n independent SDOF problems with already known solutions, and combining the result in proportion to the participation of the n^{th} mode. According to (5) the displacements can be expressed as,

$$[v] = \Sigma^N [\phi_n]\, \alpha_n\, Y_{no} \quad (15)$$

Hence the velocities are,

$$d[v]/dt = \Sigma^N [\phi_n]\, \alpha_n\, dY_{no}/dt \quad (16)$$

and the absolute accelerations are,

$$d^2[v]/dt + [1]\, d^2v_g/dt = -\, \Sigma^N [\phi_n]\, \alpha_n\, \omega_n^2 Y_{no} \quad (17)$$

Therefore for time-history analysis of a MDOF structure subjected to a ground acceleration, the solution is obtained via (15) to (17) and the appropriate SDOF solution.

4.2 Response Spectrum Analysis

If only maximum values are required, response spectra can be utilised as for the SDOF problem. Then for each mode n,

$$S_{dn} = S_{vn}/\omega_n \quad (18)$$

$$v_n = \phi_n\, \alpha_n\, S_{dn} \quad (19)$$

$$d^2v_n/dt^2 = \omega_n^2\, v_n \quad (20)$$

$$q_n = M_n\, d^2v_n/dt^2 \quad (21)$$

S_{dn} and S_{vn} are the spectral displacement and velocity respectively, and (19) to (21) are the n^{th} mode displacement, acceleration and inertia force respectively. For each dof (each mass), these quantities are then combined by statistical techniques with the most common being the SRSS combination expressed as, SRSS of the quantity = $\sqrt{(\Sigma^N \text{quantity}^2)}$. This is required since the modes of vibration for each mass do not peak at the same time.

4.2.1 Practical Application of Modal Response Spectrum Analysis to Earthquake Building Analysis

Dynamic analysis of a building using the modal analysis method as presented above is required if the building is vertically irregular in terms of mass and stiffness. A quantitative means of determining if a building has this type of irregularity is as follows:

1. Calculate the lateral inertia forces on the building at each floor as per the code ELF formula for vertical distribution of the base shear, V. The formula uses the height of each floor, h_i.
2. Calculate the storey shears.
3. Compute lateral displacements x_i at each floor under the forces from 1.
4. Substitute the x_i from 3 in place of the h_i in 1.
5. Re-calculate the story shears.
6. Compare the storey shears from 2 with those from 5 for each floor.
7. If for any floor the difference is more than 30%, then a dynamic analysis must be done.

The principle of this procedure is due to the fact that the formula is based on the ratio of floor variable to the sum of the same variable for all floors.

If it is determined that a dynamic analysis is required, the most widely used approach for design purposes, is the modal response spectrum analysis method with the main equations being (18) to (21). The overall procedure of the modal response spectrum analysis method is:

1. Model the structure as a MDOF shear building and determine the [K] and [M] matrices.
2. By using an eigenvalue solver or otherwise, obtain the modal circular frequencies.
3. By substitution in (3), determine the modal shape for each vibration mode and assemble the modal shape matrix.
4. Convert the modal frequencies from circular (ω, radians per second) to rectilinear (f, Hertz) by dividing by 2π
5. For each mode shape, and from the response spectrum, determine S_{vn} or S_{dn}. This may require conversion of the result from 4, to the modal period (T = 1/f, seconds).
6. For each mode shape determine: L_n (eq 8); the generalised mass M^*_n (9), and α_n (10).
7. For each mode shape, and for each floor per mode shape, determine the displacement (eq (19)), the acceleration (20), and the inertia force (21).
8. For each floor, combine the results of 7 for each mode shape by the SRSS combination.