CE 31B STRUCTURAL ENGINEERING

YIELD LINE METHOD

OUTLINE OF TOPICS

(3 No. Sessions at 1 hr each)

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Delivery Media: ☑ Oral ☑ Blackboard ☑ Handouts ☐ Slides/Transpar-encies ☑ Internet

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Objective: Introduction to practical limit analysis applied to slabs.

Scope/Limitations: Upper bound theorem only.

Primary Approach: Procedure-Based; Graphics-Based; Example-Based.

TOPICS

1.0 Introduction

2.0 Conventions and Assumptions

3.0 Energy Dissipation in a Yield Line

4.0 The Optimum Location of the Yield Line Pattern

5.0 Yield Line Analysis Procedure

6.0 Yield Line Analysis Example

CE 31B YIELD LINE ANALYSIS

1.0 Introduction

Yield line analysis is an analysis approach for determining the ultimate load capacity of reinforced concrete slabs and was pioneered by Johansen. The Yield Line Method is closely related to the Plastic Collapse or Limit analysis of steel frames, and is an Upper Bound or Mechanism approach.

The solution for the slab in question is presented as a load, expressed as a function of the moment (or moments) of resistance of strips of the slab, the geometry of the slab, and the location of the yield lines. Therefore, the yield line method is well suited to the reinforced concrete design objective of determining the required reinforcement. However, the method gives no information on cracking or deflection under service loads.

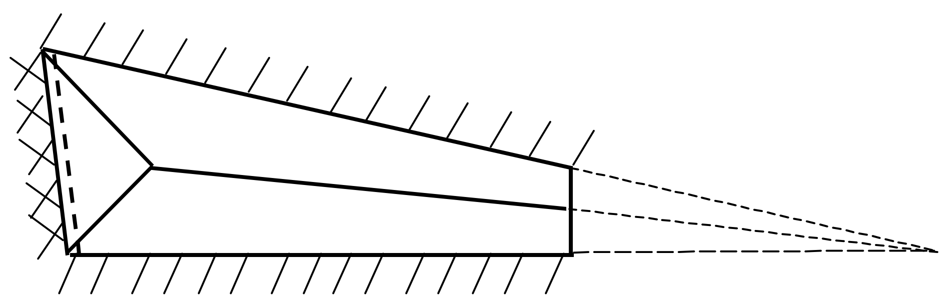
The main advantage of this method over the conventional code-based approach to slab analysis and design is its ability to cater for irregular slabs, slabs with uncommon support conditions, and slabs with uncommon loading.

Parallel to this method is the Hillerborg Strip Method which is a Lower Bound approach and is more laborious. This approach is outside the scope of this presentation but is mentioned as a source for further reading and research by the student.

The following presentation of the yield line method is introductory in that only the central concepts, and its application to simple cases are discussed.

2.0 Assumptions and Conventions

a. The slab is under-reinforced, and shear failure, bond failure and over-reinforced failure are prevented.
b. The moment-curvature relationship is idealised as the elastic-perfectly plastic curve with a long horizontal portion.
c. The assumed collapse mechanism is defined by a pattern of *yield lines* along which the reinforcement has yielded.
d. The location of the yield lines depends on the support conditions, and the loading conditions.
e. The yield lines divide the slab into several regions called *rigid regions* which are assumed to remain plane, so that all rotations take place in the yield lines only.
f. Yield lines are straight and they end at a slab boundary.
g. A yield line between two rigid regions must pass through the intersection of the axes of rotation of the two regions; the supports form the axis of rotation.



h. Yield lines are drawn in accordance with a certain convention - a solid line (———) represents a positive yield line caused by a sagging yield moment, so that the concrete cracks in tension on the underside of the slab; a broken yield line (– – – – – -) represents a negative yield line caused by a hogging yield moment so that tensile cracking occurs on the topside of the slab.

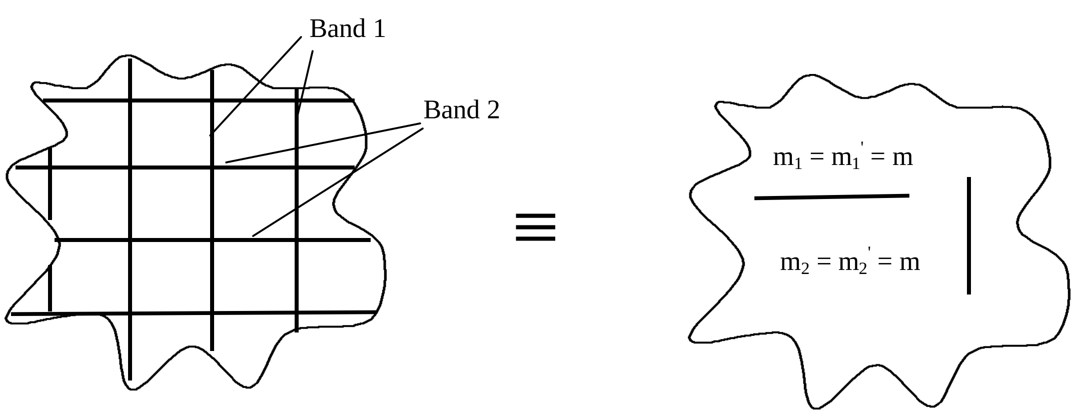
i. The convention for support conditions is as follows: single hatching represents a simply supported edge; double hatching represents a built-in edge or continuity of the slab over the support, and no hatching represents a free edge.

In the most general case for the arrangement of the slab reinforcement, the reinforcement will consist of two different bands of rebars arbitrarily inclined to each other, and arbitrarily inclined to the yield lines, where each band consists of both top and bottom reinforcement. This enables the designer to investigate a wide range of possible solutions and a wide variety of support types especially angles that define irregular slabs.

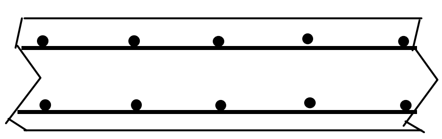
However for our purposes, we will consider only the case where the reinforcement is:

1. Of a maximum of two bands at right angles to each other
2. The two bands are of the same steel rods and distribution

In this case, the slab is said to be *isotropically reinforced*. As such, due to this reinforcement pattern, the *yield moment of resistance per unit width of the slab* is represented by the letters, *m* for the bottom steel, and *m'* for the top steel, and is the same for each band of reinforcement. Therefore, the isotropically reinforced slab looks like this:



PLAN



ELEVATION

For isotropically reinforced slabs, the reinforcement is usually parallel (and perpendicular) to the predominant support line. Also, in the conventional notation, the m lines, called the *moment axes*, are perpendicular to the reinforcement, as shown above.

Since the two layers of reinforcement at the top or bottom of the slab cannot be at the same effective depth, then strictly speaking, m_1 and m_2 are not identical. However, this is ignored in the analysis.

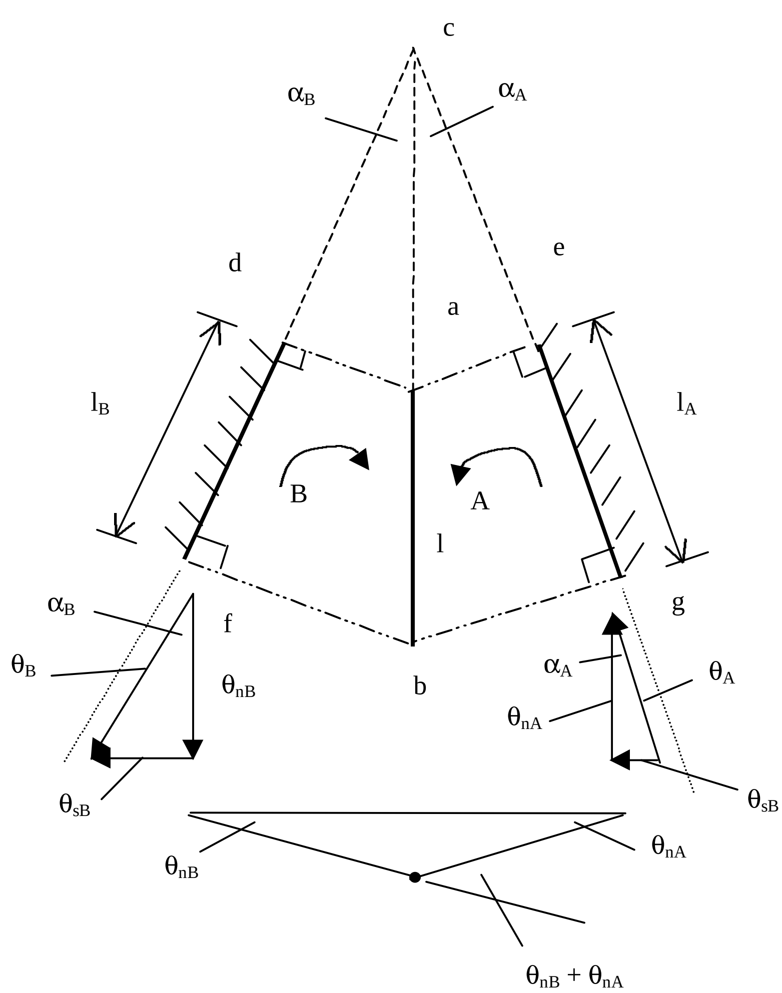
Note that in an isotropically reinforced slab, the reinforcement is nevertheless at an angle to the yield line.

3.0 Energy Dissipation in a Yield Line

Since this is an upper bound approach, it is imperative to determine the energy absorbed by the reinforcement along the yield line. By consideration of the geometry of the problem, a simple formula becomes apparent. This formula can be expressed as:

The energy dissipation in a yield line of length l due to a single band of reinforcement crossing the line, is equal to - the sum of the product of m, the projection of l onto the support axis, and the rotation of the rigid region about that axis, for the two adjacent axes.

This can be derived for any arrangement of slab and a yield line as is shown in the following example (note that the yield line is vertical):



The yield line ab divides the slab portion dfbgea into the two rigid portions A and B. A rotates θ_A about support axis eg and B rotates θ_B about support axis df. The rotations are represented by vectors following the right-hand-corkscrew rule.

If m is the moment of resistance per unit length, then the energy dissipation per unit length of the yield line

$= m\ (\ \theta_{nA} + \theta_{nB}\ )$

$= m\ \theta_A \cos \alpha_A + m\ \theta_B \cos \alpha_B$

For the length l of the yield line,

$= m\ \theta_A\ l \cos \alpha_A + m\ \theta_B\ l \cos \alpha_B$

$= m\ \theta_A\ l_A + m\ \theta_B\ l_B$

where l_A and l_B are respectively the projections of l on the axes of rotation for the rigid regions A and B. Hence, the energy dissipation for length l of the yield line is,

$= m\ \Sigma$ (projection of l on an axis) (rotation of rigid region about that axis)

4.0 The Optimum Location of the Yield Line Pattern

Remember from plastic collapse analysis of steel frames by the upper bound approach, since we have assumed a mechanism, the yield condition of collapse is not involved so there is no guarantee that the assumed mechanism is the true collapse mechanism. We must investigate other mechanisms to search for the minimum collapse load.

Likewise, for yield line analysis we have two possibilities, called the arithmetic approach, and the algebraic approach. In the arithmetic approach, we examine a series of alternative yield line patterns in search of the lowest collapse load, as is done for steel frames.

But in the algebraic approach, the yield line pattern is expressed as a function of a variable. We then use the condition that when the work equation is a maximum, the derivative of the equation with respect to the yield line position variable must be zero. We solve this derivative equation which is set to zero, for the position variable, and back substitute in the work equation. This approach will be used in the following example.

5.0 Yield Line Analysis Procedure

The following is for the case of slabs under uniformly distributed loads. The core of the procedure is the application of the following work equation:

Energy due to external loads = Energy dissipated in the yield lines

$$\Sigma\ W\ \delta = \Sigma\ m\ \theta\ l \qquad (1)$$

where W is the applied load,
δ is the deflection
m is the moment of resistance per unit length of the slab due to the rebar
l is the length of the yield line.
The summation Σ is applied for each yield line and the rigid regions about the line.

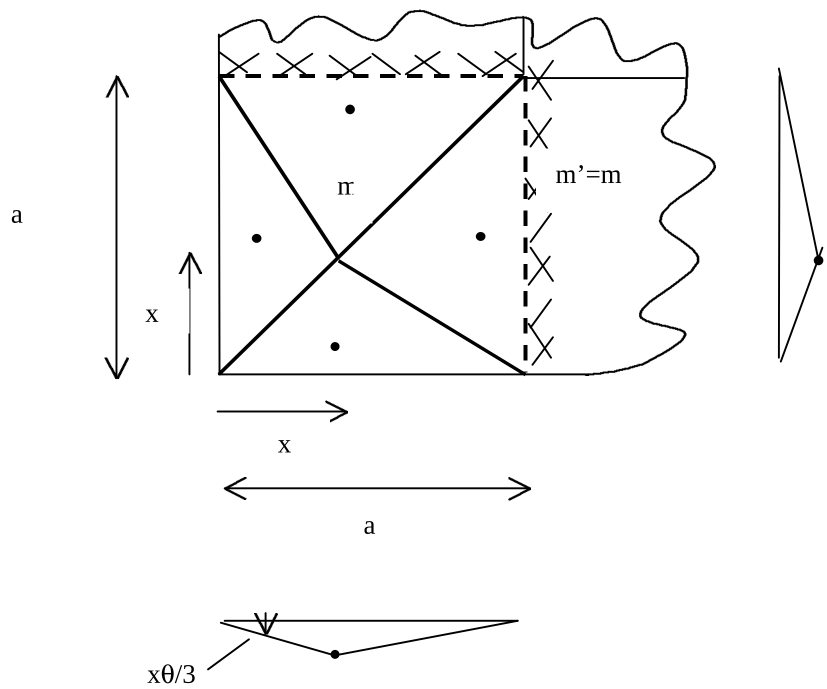
Step 1. Assume a yield line pattern.
Step 2. Since the pattern defines the locations of the rigid regions, determine the centroid and area of each rigid region
Step 3. Determine the deflection of each rigid region by considering the centroid of the region. Note that as the slab is 2-way spanning, the components of the deflections in each direction must be determined.; also, for small angles $\tan \theta = \theta$.
Step 4. Noting that W = wA, from 2 and 3 determine the left hand side of eq (1).
Step 5. Determine the right hand side of eq (1) by using the formula for the energy dissipation in a yield line.

Step 6. If the yield line pattern is not expressed as function of a position variable, then solve for w, and repeat steps 1 to 6 until no other patterns are possible, and use the lowest w as the failure load.

Step 7. If the yield line pattern is expressed as a function of a position variable, use the approach in section 4 to get the minimum w.

6.0 Yield Line Analysis Example

A 6m square corner panel of a reinforced concrete floor slab is simply supported on the outer edge on steel beams, and continuous over the interior beams. The ultimate load is 12.4 kN/m^2. Determine the required ultimate moment of resistance if the slab is to be isotropically reinforced.



Work done by the external loads is the sum of the load through the centroid of each rigid region, times the deflection of the rigid region at the centroid.

From the deflection diagrams, all the centroids have the same deflection of $x\theta/3$, hence the total external work done,

$$E = wa^2\, x\theta/3 \qquad \text{(a)}$$

The work done in the yield lines, I, from section 4 and taking the projections, we get,

$$I = 2max\theta/(a-x) + 2mx\theta + 2m(a-x)x\theta/(a-x) + 2m(a-x)\,\theta + 2mx.x\theta/(a-x) \qquad \text{(b)}$$

$$= 2m\theta a(a+x)/(a-x)$$

Equating (a) and (b) and solving, we get

$$m = wax(a-x)/[6(a+x)] \qquad \text{(c)}$$

The minimum value of m is given when $dm/dx = 0 = 6(a+x)wa(a-2x)-wax(a-x)6$, or

$$x^2 + 2ax - a^2 = 0$$

Hence $x = 0.414a$.

Substituting in (c) and for $a = 6m$, we get

$m = 0.0286\ wa^2 = 12.76$ kNm/m