2.2.2.1 Frequency Domain

Each of the response solutions presented in section 2.2.2.1 above are functions of time, hence the solutions are said to be in the time domain. Time domain solutions do not indicate which frequencies are most significant in terms of contribution to the overall response. Such information will indicate to the engineer which natural frequencies to avoid in order to minimize amplification due to resonance effects.

Alternatively, if the solution is presented in the frequency domain, in which case, the loading or response is represented graphically by a spectrum, the ordinates immediately indicate the most significant frequencies.

Another advantage of solution in the frequency domain is that when the spectrum is determined, the peak response is readily determined as the product of the corresponding peak load, and the transfer function or frequency response function.

In the following sections, frequency domain solutions are presented for periodic, non-periodic, and random loading.

2.2.2.2.1 Periodic

Noting the exponential from of the Fourier Series as indicated in eqn (2.9), eqn (2.51) can be re-written as,

$$M\frac{d^2v}{dt^2}+C\frac{dv}{dt}+Kv=\sum_{n=-\infty}^{\infty}P_n\,e^{in\omega' t} \tag{2.54}$$

From the Principle of Superposition (as the system is linear), the response has the form,

$$v=\sum_{n=-\infty}^{\infty}X_n\,e^{in\omega' t} \tag{2.55}$$

Substituting in (2.54),

$$M\sum_{n=-\infty}^{\infty}X_n(in\omega')^2e^{in\omega' t}+C\sum_{n=-\infty}^{\infty}X_n(in\omega')\;e^{in\omega' t}+K\sum_{n=-\infty}^{\infty}X_ne^{in\omega' t}=\sum_{n=-\infty}^{\infty}P_ne^{in\omega' t}$$

Collecting terms at each frequency,

$$(-n^2\omega'^2M+iCn\omega'+K)X_n=P_n \tag{2.56}$$

Hence,

$$X_n = H_nP_n \tag{2.57}$$

H_n is called the transfer function or frequency response function (FRF). Eqn (2.57) indicates that the output is simply a constant times the input. It is therefore also called input-output relationship.

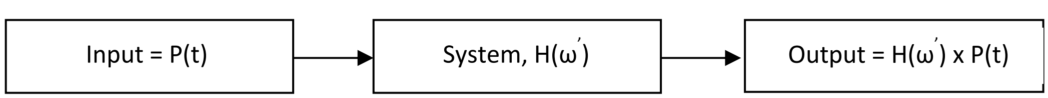

Figure 14

Furthermore, since H_n is a function of the input load frequencies, the system acts like a filter because in the output, the frequency content is altered. This term – filter, is from the application of dynamics to the field of electrical engineering called signal processing.

$$H_n = \frac{1}{-n^2\omega'^2 M + iCn\omega' + K} = \frac{(-n^2\omega'^2 M + K) - i(Cn\omega')}{(-n^2\omega'^2 M + K)^2 + (Cn\omega')^2} \tag{2.58}$$

Note that H_n is a complex number. However, if the system is undamped, it can be shown that it becomes a real number. Eqn (2.58) can be re-written in terms of the frequency ratio $r = \omega'/\ \omega_n$, and the damping ratio $\xi = C/2M\omega_n$.

$$H_n = \frac{1}{K[(1-n^2r^2) + i(2nr\xi)]} = \frac{(1-n^2r^2) - i(2nr\xi)}{K[(1-n^2r^2)^2 + (2nr\xi)^2]} \tag{2.59}$$

Therefore, the phase is given by,

$$\phi_n = \tan^{-1}\frac{\text{Im}\ (H_n)}{\text{Re}\ (H_n)} = \tan^{-1}\left(\frac{2nr\xi}{1-n^2r^2}\right) \tag{2.60}$$

Given that the modulus of a complex number is associated with its phase, eqn (2.55) can be expressed in the modulus-phase form with respect to H_n. This requires the complex conjugate of H_n, denoted as, H_n^* and the relationship $|H_n| = [H_n\, H_n^*]^{1/2}$, where $|H_n|$ means the modulus of H_n. Therefore,

$$|H_n| = \frac{1}{K[(1-n^2r^2)^2 + (2nr\xi)^2]^{1/2}} \tag{2.61}$$

$$v = \sum_{n=-\infty}^{\infty} |H_n|\, P_n\, e^{(in\omega' t + \phi n)} \tag{2.62}$$

Likewise, P_n in eqn (2.62) can be expressed in modulus-phase form. Hence (2.62) can be written as

$$v = \sum_{n=-\infty}^{\infty} |H_n|\, |P_n|\, e^{(in\omega' t + \phi n + \alpha n)} \tag{2.63}$$

where α_n is the phase angle associated with P_n.

$|P_n|$ is determined from eqn (2.13) in section 2.1.1.2 above. Recall that P_n are the Fourier coefficients when the Fourier Series is in exponential (i.e. complex number) form.

2.2.2.2.2 Non-Periodic

As observed in the last section, when the load is periodic, frequency domain analysis is based on the Fourier Series. When the load is non-periodic or arbitrary, frequency domain analysis is based on the Fourier Integral. The characteristics of the Fourier Integral are discussed in section 2.1.2. Hence the displacement solution is

$$v(t) = \int_{-\infty}^{\infty} H(f)P(f)e^{i2\pi ft}dt \tag{2.64}$$

Note in this case the replacement of the circular frequency ω with the rectilinear frequency f. This is because of its more practical use. H(f) is thus given by

$$H(f) = \frac{1}{K[(1-R^2) + i(2R\xi)]} \qquad (2.65)$$

where $R = 2\pi f/\omega_n$ and f is with respect to the forcing function.

Response Spectrum

It is noted in section 2.2.2.1.4 above, that in the time domain the convolution integral can be used to calculate response spectra. In the frequency domain, eqns (2.63) and (2.64) are also very useful for this purpose because, for example, if the positive real values of $|H||P|$ are plotted against frequency, the resulting graph is the amplitude response spectrum. This approach is quite powerful if numerical methods are used to obtain the Fourier coefficients, namely the Fast Fourier Transform (FFT), discussed in subsequent sections.