4 Dynamics of Continuous Systems – Beam Under Lateral Load

In the previous chapters the structure is modeled as a lumped-parameter system and this idealization is clearly far removed from the actual physical system which is continuous in the three dimensions of space. It is possible to capture the continuous nature of physical reality for structural dynamic analysis and such models are called distributed-parameter or continuous systems.

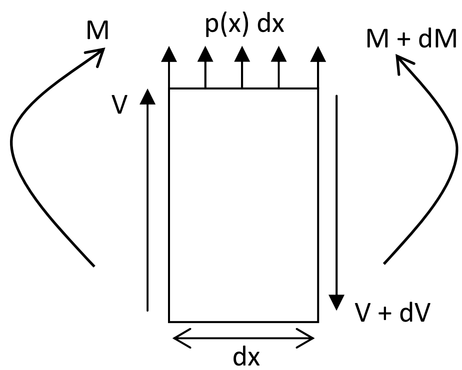
The governing equations of systems modeled as continuous systems are much more difficult to solve mathematically though in the cases when solution is possible, the solutions are called exact solutions. It is for this reason, and the reasonably good accuracy obtained, that lumped-parameter models are more common in practice.

It is noteworthy to mention that the method of Finite Element Analysis (FEA) is a go-between in terms of distributed and lumped parameter modeling. Its wide range of applicability and considerable accuracy makes this the method most used for advanced dynamics studies. FEA is a numerical method hence requires extensive computational resources for its use. The mechanical and computational sophistication of FEA implies the need for specialist training for its effective use.

In this section, the dynamics of a continuous beam of constant section under lateral load is considered. Even for this most simple of cases, response solution for external dynamic loading is beyond the scope of the presentation. Furthermore, a simplification is made by neglecting the contribution of shear deformation, rotary inertia, and damping. The presentation considers loading due to the inertia associated with the distributed self-weight of the beam. In effect, this is the study of the free vibration of the beam.

4.1 Modeling and Analysis

Recall the free body diagram of the basic engineer's beam theory or Euler's beam theory as shown below. This is for an infinitesimal element along the beam's length. M, V, and p(x) are the moment, shear, and inertial load per unit length respectively.



For vertical equilibrium,

$$dV - p(x)\,dx = 0 \tag{4.1}$$

Taking moments about the right face,

$$dM - Vdx - 0.5p(x)(dx)^2 = 0 \tag{4.2}$$

From (4.1),

$$\frac{dV}{dx} = p(x) \tag{4.3}$$

From (4.2), ignoring higher order effects,

$$\frac{dM}{dx} = V \tag{4.4}$$

Combining (4.3) and (4.4),

$$\frac{d^2M}{dx^2} = \frac{dV}{dx} = p(x) \tag{4.5}$$

The constitutive relation for a beam is given by,

$$M = EI\frac{d^2y}{dx^2} \tag{4.6}$$

where y is the beam's deflection and EI is its flexural rigidity. Substituting in (4.5) and considering EI as constant along the beam's length,

$$EI\frac{d^4y}{dx^4} = p(x) \tag{4.7}$$

For harmonic motion, and as damping is neglected, the inertia force per unit length due to self-weight must equal $\rho\omega^2 y$, where ρ is the beam's mass per unit length. Substituting in (4.7),

$$EI\frac{d^4y}{dx^4} - \rho\omega^2 y = 0 \tag{4.8}$$

$$\text{Let } \beta^4 = \frac{\rho\omega^2}{EI} \tag{4.9}$$

Hence,

$$\frac{d^4y}{dx^4} = \beta^4 y \tag{4.10}$$

Therefore, (4.10) is the equation of free vibration for the beam.
From (4.9) the free vibration or natural frequencies are given by,

$$\omega_n = (\beta_n L)^2\sqrt{\frac{EI}{\rho L^4}} \tag{4.11}$$

where L is the beam's length. It can be shown that the solution of (4.10) is given by,

$$y = A\cosh\beta x + B\sinh\beta x + C\cos\beta x + D\sin\beta x \tag{4.12}$$

The constants A, B, C, and D are determined by considering the conditions at the ends of the beam. There are an infinite number of degrees of freedom for a continuous system. However, the natural frequencies for the first three modes are as shown in the following table for various end conditions. These values are for $(\beta_n L)^2$.

Configuration	**1st Mode**	**2nd Mode**	**3rd Mode**
Simply-supported	9.87	39.5	88.9
Cantilever	3.52	22.0	61.7
Free-Free	22.4	61.7	121.0
Clamped-Clamped	22.4	61.7	121.0
Clamped-Hinged	15.4	50.0	104.0
Hinged-Free	0	15.4	50.0

4.2 Approximate Determination of Fundamental Natural Frequency

The Rayleigh method for the calculation of the approximate first or fundamental natural frequency of an MDOF system was previously considered. The concept can be extended to the continuous beam as well. In such a case, the Rayleigh frequency is given by,

$$\omega_1^2 = \frac{\int_0^L EI(x)\left[\frac{d^2 y_a}{dx^2}\right]^2 dx}{\int_0^L \rho A(x) y_a^2 dx} \tag{4.13}$$

where $y_a = y_a(x)$ which is the assumed deflection profile of the beam.

Note that in (4.13) ρ is the mass density of the beam, and it caters for a beam with possible varying cross-section along its length. y_a is typically chosen to be the same as the beam's deflection profile under static load. Therefore, standard texts can be consulted to obtain formulae for the deflection.