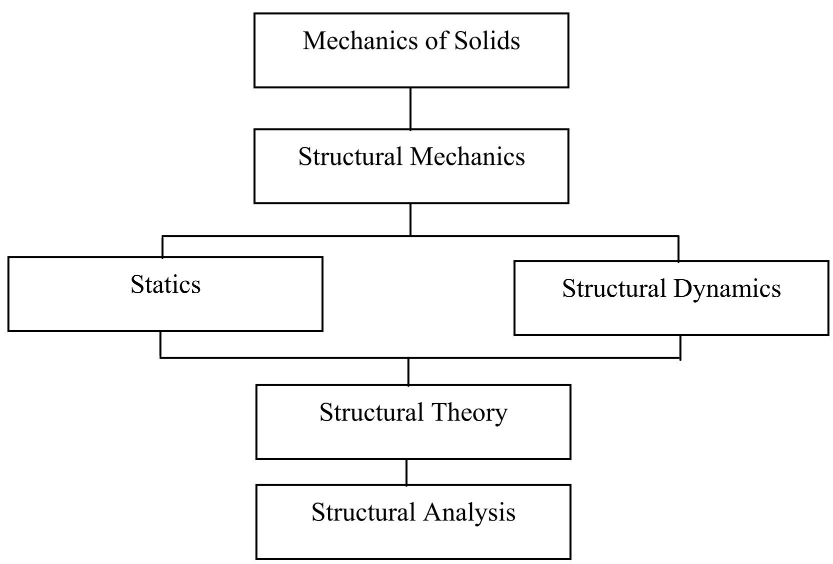
# Introduction – What is "Mechanics of Solids?"

Mechanics (Greek *Μηχανική*) is the branch of physics concerned with the behaviour of physical bodies when subjected to forces or displacements, and the subsequent effect of the bodies on their environment. (Wikipedia).

Mechanics of Solids is the computation of the internal stresses and deformations within **deformable** solid bodies when subjected to external forces.

Mechanics of Solids is therefore a necessary part of Civil Engineering design in which given the calculated stresses, the amounts of materials required to resist these stresses are determined based on economics, sustainability, and aesthetics considerations.

In the general context of Civil and Environmental Engineering, Mechanics of Solids has a number of interrelated sub-areas as follows, and is therefore an umbrella term covering these areas:




In the actual usage of Mechanics of Solids, the solid bodies of interest fall into 2 groups – structures composed of linear elements (e.g. steel frameworks), and "bulk masses" (e.g soils). In our diagram above both are referred to when we use the terms "structural", so "Mechanics of Solids" is virtually synonymous with "Structural Mechanics".

"Statics" is that aspect of the Mechanics of Solids in which the focus is on when the solid body is in a state of uniform motion or at rest, and under a set of forces which do not change and are therefore static. In "Statics" we apply Newton's Third Law to determine the internal stresses and deformations.

"Structural Dynamics" is that aspect of Mechanics of Solids in which the focus is on when the solid body is under a set of forces that change in time and sufficiently quickly that inertia forces also act on the solid body. Therefore in "Structural Dynamics" we apply Newton's Second Law to determine the internal stresses and deformations.

Both "Statics" and "Structural Dynamics" make use of "Structural Theory" and "Structural Analysis". In "Structural Theory" mathematical theories are developed which account for the means by which a solid body moves internally and develops its stresses. Being mathematical theories, they result in theorems or truths about the inner workings of solids. In "Structural Analysis", we make use of the theorems of "Structural Theory" to develop methods of analysis that directly result in the determination of the internal stresses and deformations of the solid (which is the main objective of Mechanics of Solids, or Structural Mechanics).

In our Level I UWI Mechanics of Solids course, we familarise ourselves with the most basic types of solid bodies under the most basic sets of forces, using the most basic structural theories and analyses which enable us to calculate the internal stresses and deformations in these bodies.

As a student advances through the Civil and Environmental Engineering program he or she is progressively exposed to more complex solid bodies of interest and under more complex loads. The main objective of determining the internal stresses and deformations of solids remains the same, but the name "Mechanics of Solids" changes to "Structural Mechanics" at Level 2, and "Structural Analysis" in Level 3, in order to handle the increased complexity in a more structured fashion.

When conducting library searches, the student may encounter such terms as "Mechanics of Materials" or "Strength of Materials". These topics are parallel to Mechanics of Solids but tend to cover solid bodies of interest of mechanical engineers, such as gears, springs, etc.

In Section 1, we examine a systems view of solid bodies of interest in Civil and Environmental Engineering.

# Units of Measurement

In our Mechanics of Solids course, the following Units of Measurement are typically employed for the special quantities of interest:

| QUANTITY | UNIT |
|---|---|
| Force | Newton (N) |
| Moment | Newton-meter (Nm) |
| Displacement | m |
| Rotation | radian |
| Slope | radian |
| Curvature | $m^{-1}$ |
| Modulus of Elasticity | Pascal, Pa ($N/m^2$) |
| Moment of Inertia | $m^4$ |
| Section Modulus | $m^3$ |
| Stress | Pascal, Pa ($N/m^2$) |
| Strain | Dimensionless |
| Shear Modulus | Pascal ($N/m^2$) |
| Torque | Newton-meter (Nm) |

These units are commonly used with the following prefixes:

| UNIT | PREFIX |
|---|---|
| Newton (N) | Kilo (kN)<br>Mega (MN)<br>Giga (GN) |
| Newton-meter (Nm) | Kilo (kNm)<br>Mega (MNm)<br>Giga (GNm) |
| m | Milli (mm)<br>Centi (cm) |
| Pascal ($N/m^2$) | Kilo ($kN/m^2$ or kPa)<br>Mega ($MN/m^2$ or MPa = $N/mm^2$) |

Kilo = $10^3$; Mega = $10^6$; Giga = $10^9$; Milli = $10^{-3}$; Centi = $10^{-2}$

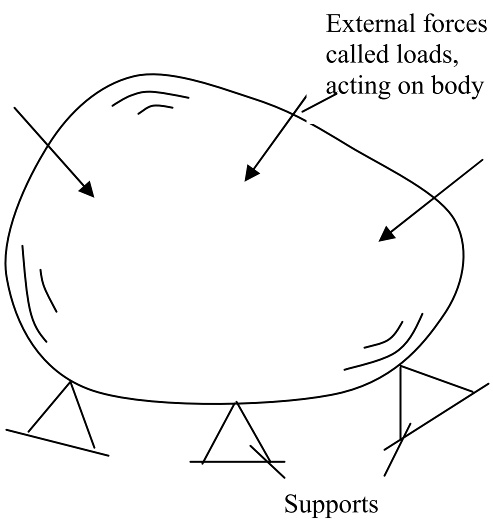
# Convention Used in This Book

To assist the student synthesize the ideas presented in the following sections, as well as provide the critical references used in Mechanics of Solids problem-solving, a convention is used throughout this book.

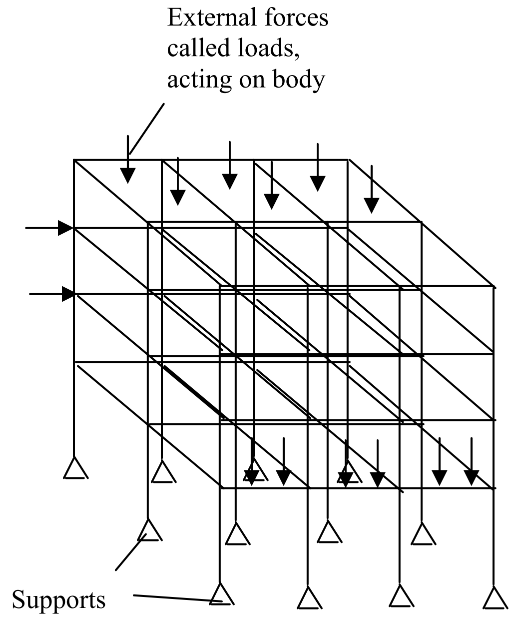
This convention is the use of numbered summary phrases called" Important Facts". The phrase begins with the abbreviation IF followed by the hash character then a number, and all letters in bold font. For example,

**IF # 1: Civil Engineers shape the environments of society.**

## 1.0 TYPES OF SOLIDS – A SYSTEMS VIEW OR SEQUENCE OF IDEALIZATIONS




(a) Bulk Mass resisting applied loads




(b) Framework resisting applied loads

Fig. 1.1 The Two Main Types of Solids

In the built environment of society, there are really only 2 types of solids - the Bulk Mass, and the Framework. Both types of solids are three-dimensional (3D). Examples of these are shown in Fig. 1.1. The framework can exist in 2 forms, the one shown in Fig. 1.1b - called a Space Frame, and the one shown below, called a Space Truss.

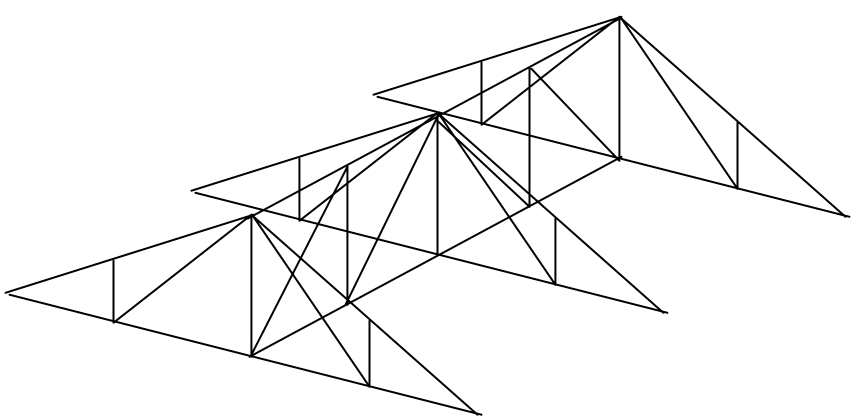


Fig. 1.2 Example of a Space Truss

Though every solid is 3D in reality, by virtue of their geometrical forms, we represent the form in total as the Bulk Mass and the Framework. Examples of the bulk mass form in civil engineering are the pyramids of Egypt, and the soil mass beneath a structure. Examples of 3D frameworks are the skeletons of buildings for space frames, and bridges for space trusses.

However, there is a much more important reason why we represent forms in these ways, and other ways we will soon discuss. In the calculation of the internal stresses and deformations the engineer desires to do so in the shortest time possible. This is because the calculations are part of the service being provided by the engineer to a client, so the faster the calculations are performed, decisions can be made sooner, and more clients can be serviced in a given time. With this time constraint, to make the calculations easier, the engineer represents the solid as a set of simpler solids and does the calculations for that simpler form instead. This process of representing the actual solid by a simpler one is called **Idealization.** To simplify things usually requires that the engineer make assumptions to ensure that the idealization does not result in calculations that are too inaccurate. The simplified calculations are compared to more complex calculations, or to test results in order to show that the simplified calculations are acceptable.

**IF # 2 : To simplify calculations in Mechanics of Solids, the engineer idealizes the solid by representing it as a simpler solid (e.g. from a 3D form to a set of 2D forms).**

In the reminder of this section, we present the idealizations made in Mechanics of Solids for the 3D frameworks of space frames and space trusses, and the important characteristics of these simplifications.

It will be seen that the idealization follows a sequence from the more complex 3D form, to 2D forms, then to the individual member or element, then cross-sections of the element. Since these are all parts of the one original 3D form, it is called a "Systems View" of that form.

a. <u>2D Components (of the 3D):</u>

1. <u>The Plane Frame</u>

Remembering that in Mechanics of Solids our objective is the calculation of the stresses and deformations in the solid, if we have the case of a spaceframe, how can be simplify the calculations? Returning to the space frame of Fig. 1.1, the spaceframe can be considered as sets of 2D frames, called Plane Frames, in each of the 2 directions in plan of the structure. So instead of analyzing the whole structure, we analyze the plane frames only.

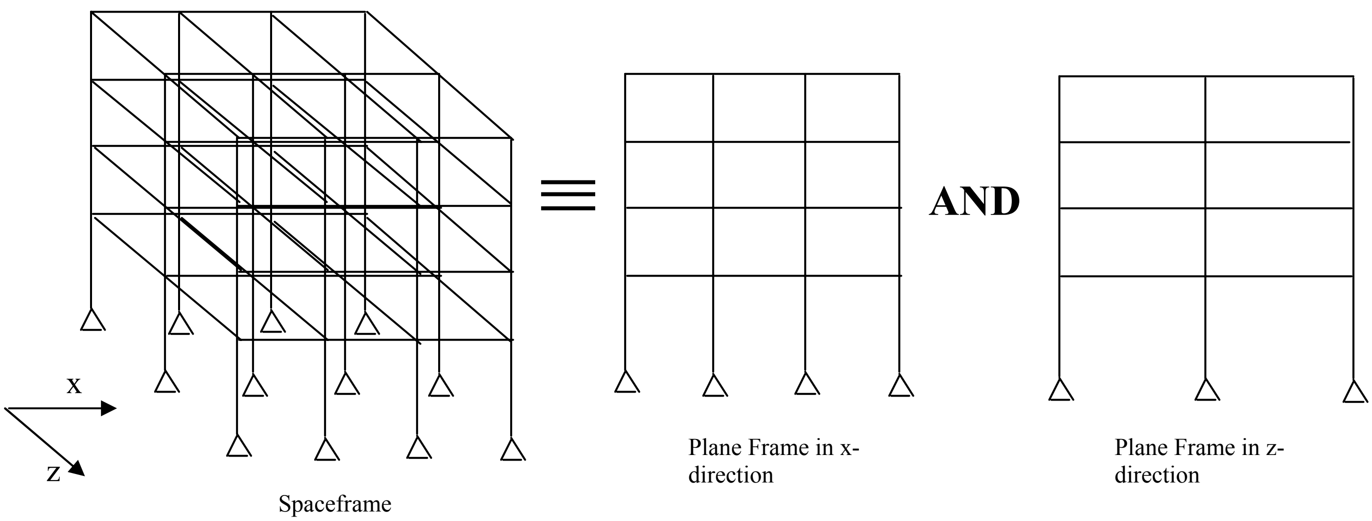



Fig. 1.3 Idealizing a Space Frame as 2 Plane Frames

2. <u>The Plane Truss</u>

As for the case of the space frame, the engineer can consider the space truss as sets of 2D trusses, called Plane Trusses, in each of the 2 directions in plan of the structure. So instead of analyzing the whole structure, we analyze the plane trusses only.

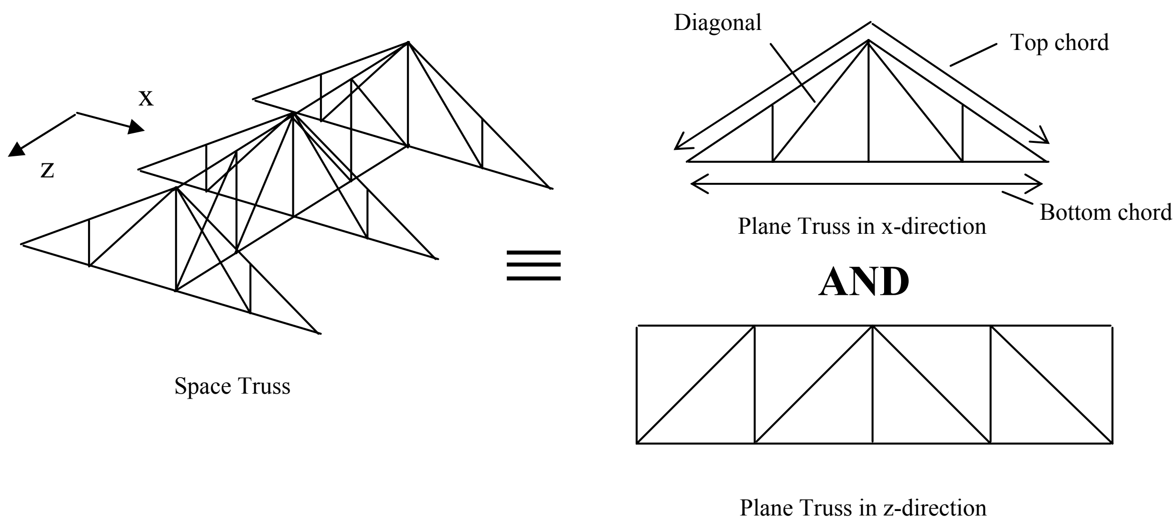



Fig. 1.4 Idealizing a Space Frame as 2 Plane Trusses

A plane truss is characterized by its having diagonal members between the ends of vertical members as shown above. The top edge of a truss is called the “top chord”, and the bottom edge, the “bottom chord”. A truss is not necessarily triangular or rectangular, and can have any number of vertical members hence diagonals. Some types of trusses are named after their inventors such as the Warren, and Pratt trusses each of which is preferred in certain situations.

b. 1D Components (of the 2D):

1. The Continuous Beam and Column

We can continue the simplification of space frames and consider the plane frame, which is 2-dimensional (2D) as a set of "Continuous Beams", and "Columns" (also called "Stanchions" if they are of steel), which are 1-dimensional (1D) since their form is completely defined by their length only.

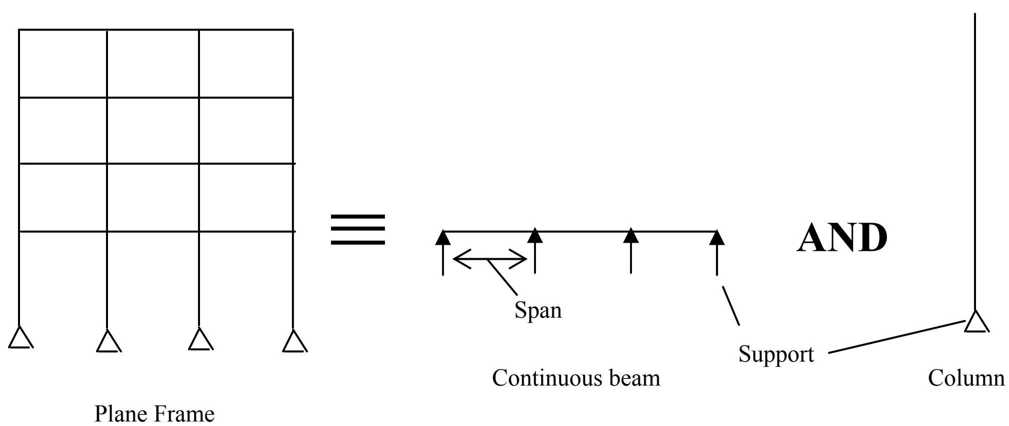



Fig. 1.5 Idealizing a Plane Frame as Continuous Beams and Columns

For the plane frame shown in Fig. 1.5 the continuous beam has 3 spans. A continuous beam can have any number of spans which can be of varying length each – it depends on the plane frame we are trying to represent. The continuous beam is so-called because as you go from one end of the beam to the other, the beam continues over the supports. Of course, the supports in the original form (i.e. the plane frame), are really the columns but we have taken these out to analyze them separately. However, we must represent their effect on the beam, which is that they act as supports of the beam.

2. The Strut and Tie

Continuing the simplification of the space truss, each member of a plane truss, which is 2-dimensional (2D), is called a strut or a tie, which is 1-dimensional (1D) since their form is completely defined by their length only.

A strut is a member that, for a given set of external forces, is under a set of internal forces at its ends trying to reduce the length of the member (i.e. a compressive force). For a tie, the forces at the ends are trying to increase the length of the member (i.e. a tensile force).

At this point, the question may be asked of the difference between continuous beams/columns, and struts/ties in Mechanics of Solids since they are both 1D members. The reason is that the internal forces of continuous beams/columns are always such as to cause the member to bend, but for the struts/ties the internal forces do not cause the member to bend. This affects the ways these elements are analyzed, as we shall see in the other sections of this book.

c. 0D Components (of the 1D):

1. The Cross-Section of Infinitesimal Length

We saw in preceding sections that for the practical reason of simplifying the calculations in Mechanics of Solids, a sequence of idealizations of the solid called the space framework is made. This resulted in the substitution of the 3D framework by a set of 1D members. However, since we are ultimately concerned with determining the stresses and deformations **within** the solid, one further idealization is required.

The 1D member can be considered composed of a collection of cross-sections from one end of the length to the other. A cross-section is planar hence has 2 dimensions, but if the 1D element is to be considered a collection of sections, a section must have a dimension in the direction of the length of the member. This dimension is of infinitesimal length.

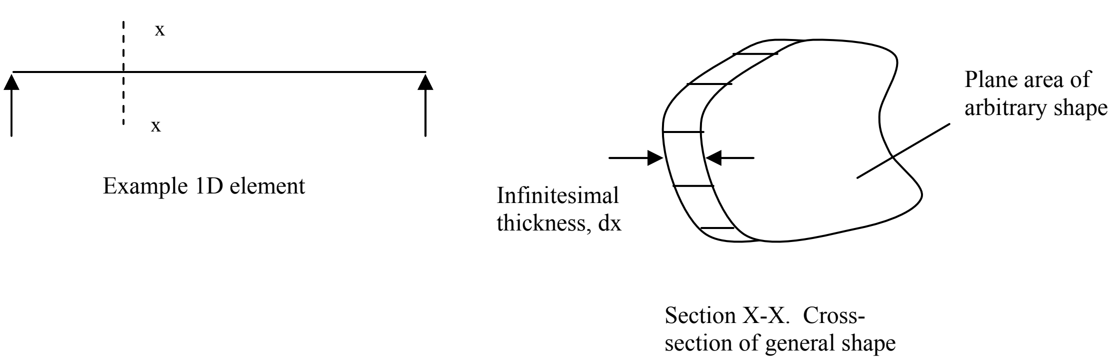



Fig. 1.6 Idealizing a 1D Member as Set of Sections

Since the section is of infinitesimal thickness then for consistency with our nomenclature, it is considered zero-dimensional.

## 2. Geometrical Properties of Sections

Now that we have represented the original 3D solid framework by a set of 1D elements comprised of sections of infinitesimal cross-sections we can now state the important fact that:

**IF # 3: For practical calculations in Mechanics of Solids, the engineer typically performs the calculations for 1-dimensional members and their sections as an idealization of the original three-dimensional solid. All the calculations of stress and deformation are for a certain section of the 1D member.**

The properties of the cross-section are therefore of central importance in the calculation of stress and deformations in solids of practical interest. These properties are geometrical properties, sometimes called "mass properties" of the section. The most important of these are: area, centroid, moment of inertia, and from these, the section modulus, and the radius of gyration.

### 2.1 The Centroid or Centre of Area

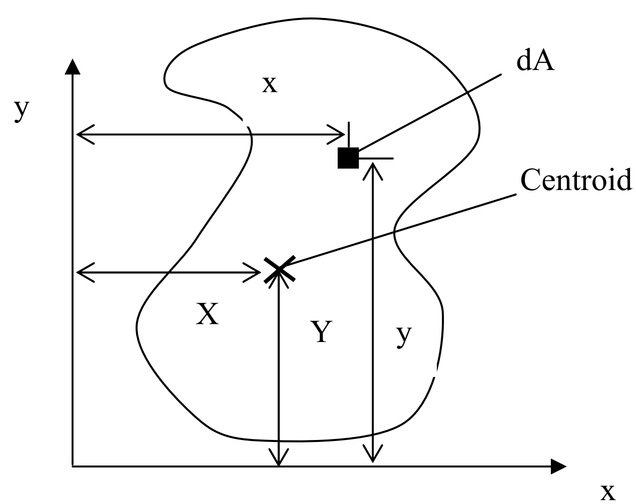



Fig. 1.7 Section of Arbitrary Shape

Referring to Fig. 1.7, consider a section of arbitrary shape. The aim is to determine the coordinates of the centroid, X and Y. For a coordinate system x-y, an infinitesimal portion of the section dA is located x from the y-axis, and y from the x-axis. If the shape is of area A, then the product of A and the distance of the centroid from the x-axis, Y, equals the sum of the product of all dA's and their distance from the x-axis, y. Hence,

$$AY = \int y.\, dA \qquad (1.1)$$

Likewise for the distance of the centroid from the y-axis, X,

$$AX = \int x.\, dA \qquad (1.2)$$

Equations (1.1), (1.2) are for general shapes. In practical situations, such as for I, C, L, and T-shaped sections, the section is composed of simple rectangular sections combined together. In such cases, we can replace the integrals of (1.1), (1.2) with summations. Hence for such sections we can express (1.1), and (1.2) as,

$$AY = \sum_{1}^{n} A_i y_i \qquad (1.3)$$

where n is the number of simple elements comprising the section and "i" is the i th element..

$$AX = \sum_{1}^{n} A_i x_i \qquad (1.4)$$

Hence to calculate the centroid of practical sections, choose any convenient location for a x-y coordinate system then use equations (1.3) and (1.4). Note that the centroid need not lie within the solid body of the section (e.g. for L and C shapes).

**IF # 4: To calculate the centroid of practical sections, choose any convenient location for the x-y coordinate system then use** $AY = \sum_{1}^{n} A_i y_i$ **and** $AX = \sum_{1}^{n} A_i x_i$ **to determine the coordinates of the centroid X,Y. The centroid does not always lie within the solid body of the section.**

## 2.2 The Moment of Inertia, Section Modulus, Radius of Gyration

Referring to Fig. 1.7, the moment of inertia of the section about the x-axis, termed as "$I_{xx}$", and about the y-axis, $I_{yy}$, are respectively defined as,

$$I_{xx} = \int y^2 dA \tag{1.5}$$

$$I_{yy} = \int x^2 dA \tag{1.6}$$

The moment of inertia is also known as the "second moment of area".

Note that from (1.5) and (1.6), the values for the moments of inertia depend on where we place the x-y coordinate system. In engineering calculations to calculate the moments of inertia the centroid of the section is first determined then the x-y coordinate system placed at the centroid.

Example 1.1:

Calculate the $I_{xx}$ and $I_{yy}$ for a rectangular section of vertical and horizontal dimensions d and b respectively.

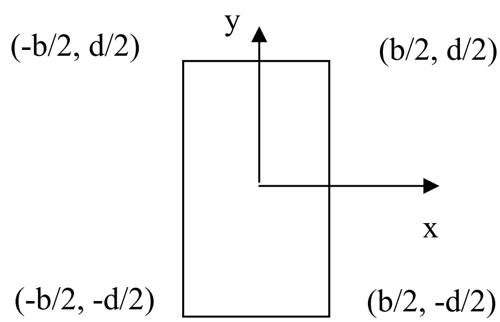



Fig. 1.8 Section of Arbitrary Shape

Fig. 1.8 shows the section with the x-y coordinate system placed at the centroid, and the coordinates of the corners also shown.

From (1.5),

$$
\begin{aligned}
I_{xx} &= \int y^2 dA \\
&= \int y^2 b dy = b \int y^2 dy \\
&= b\left[y^3/3\right]_{-d/2}^{d/2} \\
&= b\left[((d/2)^3 - (-d/2)^3)/3\right] \\
&= b\left[((d^3/8) - (-d^3/8))/3\right] \\
&= bd^3/12
\end{aligned}
\tag{1.7}
$$

Likewise,

$I_{yy} = db^3/12$ (1.8)

<u>The Subtraction Method</u>

(1.5) and (1.6) also imply that if a certain condition is met, the $I_{xx}$ (and/or $I_{yy}$) of more complex but practical shapes can be calculated by applying them to the extremities of the section including the spaces, but then subtracting the $I_{xx}$ (and $I_{yy}$) for the spaces. It is important to remember that this special case is that the relevant centroidal axis of the spaces must be along the same line as the centroidal axis of the section. Hence this subtraction method can be used for hollow circular or rectangular sections, but only for the $I_{xx}$ for I-sections that are symmetrical about the centroidal x-axis.

<u>Example 1.2:</u>

Calculate the $I_{xx}$ for the I section shown in Fig. 1.9. The top and bottom flanges and the web are all 16 mm thick. The section is 355 mm deep and 175 mm wide. The centroidal axes of the section are also shown.

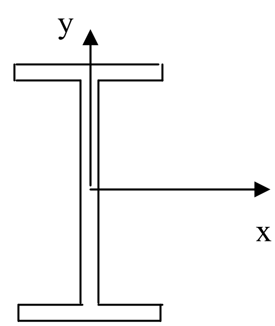



Fig. 1.9 I-section

The extremities of the section contain 2 rectangular spaces, and the centroidal x-axis of these spaces are along the centroidal x-axis of the section. Threfore, the subtraction method can be used to determine the $I_{xx}$.

As the $I_{xx}$ of a rectangle is $bd^3/12$,
$I_{xx}$ of the rectangle including the spaces is $175x355^3/12 = 6.524x10^8$ $mm^4$.
The dimensions of each space within the rectangle are 323 mm (i.e. 355-2x16) deep, and 79.5 mm (i.e. (175-16)/2) wide.

Hence the $I_{xx}$ of each space = $79.5x323^3/12 = 2.233x10^8$ $mm^4$.

Hence $I_{xx}$ for the I-section = (6.524 – 2x2.233) x$10^8$ = 2.058 x$10^8$ $mm^4$.

**IF # 5: The subtraction method of calculating $I_{xx}$ will only give correct results for hollow rectangular sections, hollow circular sections, or symmetrical I sections.**

The Parallel Axis Theorem Method

When the subtraction method cannot be used, the "parallel axis theorem" can be used to determine $I_{xx}$ or $I_{yy}$.

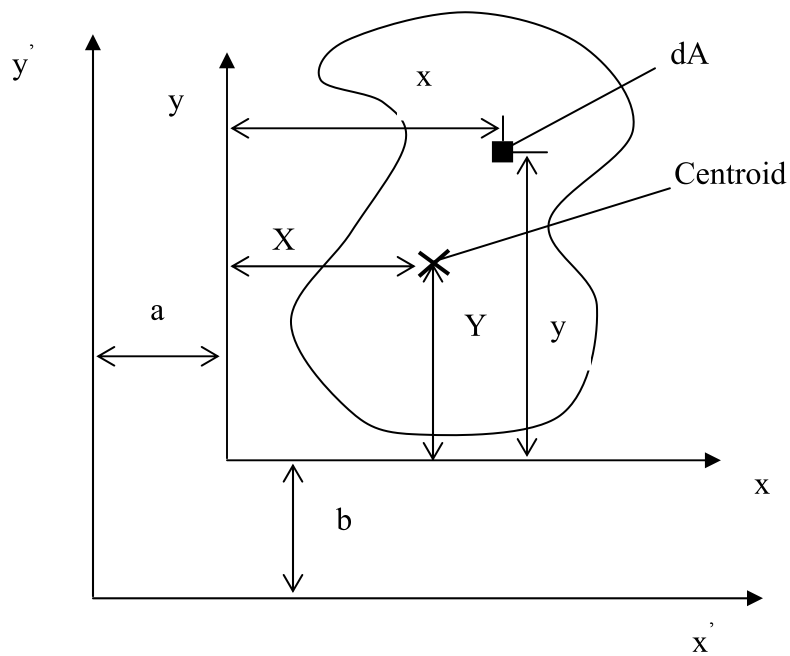



Fig. 1.10 Moments of area about parallel axes

Consider Fig. 1.10. About the x' axis, (1.5) becomes,

$$I_{x'x'} = \int (y+b)^2 dA$$

$$= \int y^2 dA + 2b\int y \ dA + \int b^2 dA$$

If the x-y coordinate system is placed at the centroid, the middle term equals zero, hence,

$$I_{x'x'} = \int y^2 dA + \int b^2 dA = I_{xx} + b^2 A \qquad (1.9)$$

Likewise,

$$I_{y'y'} = I_{yy} + a^2 A \qquad (1.10)$$

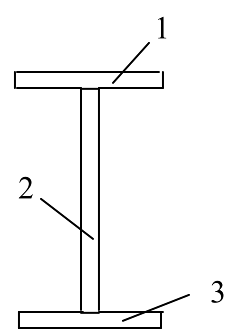
(1.9) or (1.10) is applied by first determining the centroid of the overall area, then breaking up the section into parts each with a known formula for $I_{xx}$. Then (1.9) can be written as,

$$I_{x'x'} = \sum_{1}^{n} (I_{xx,i} + b_i^2 A_i) \qquad (1.11)$$

where n is the total number of parts, i is the "i" th part, $b_i$ is the vertical distance from the centroid of the ith part to the centroid of the overall area, and $A_i$ is the area of the ith part.

<u>Example 1.3:</u>

Re-calculate Example 1.2 but using the parallel axis theorem method.




The section is broken up into three rectangular sections as shown. We already know that in this case the centroid is at the intersection of the axes of symmetry of the overall section.

Applying (1.11) we get,

$I_{xx} = 175 \text{x} 16^3/12 + (355/2 - 16/2)^2 \text{ x } 175\text{x}16 +$ (this line for part 1)
$16\text{x}(355\text{-}2\text{x}16)^3/12 +$ (this line for part 2)
$175\text{x}16^3/12 + (355/2 - 16/2)^2 \text{ x } 175\text{x}16$ (this line for part 3)
$= 2.058\text{x}10^8 \text{ mm}^4$

**IF # 6: The parallel axis theorem method of calculating $I_{xx}$ is given by**

$I_{x'x'} = \sum_{1}^{n} (I_{xx,i} + b_i^2 A_i)$. **It may be used if the conditions for using the subtraction method are not met and is very convenient if the $I_{xx}$ for the parts can be determined from formulae.**

<u>Section Modulus</u>

The section modulus about the centroidal x-axis is termed as $S_x$ and is given by,

$S_x = I_{xx}/Y$ (1.12)

Likewise,

$S_y = I_{yy}/X$ (1.13)

Y and X are the maximum distances from the centroidal x and y-axes respectively. The section moduli are very important in the calculation of the internal bending stresses on sections.

<u>Radius of Gyration</u>

The radius of gyration about the centroidal x-axis is termed as $r_x$ and is given by,

$r_x = \sqrt{(I_{xx}/A)}$ (1.14)

Likewise,

$r_y = \sqrt{(I_{yy}/A)}$ (1.15)

A is the area of the section. The radius of gyration is very important in the study of columns and struts.

d. <u>The Supports and Joints – Where Components Meet</u>

Though not a part of a solid body, all solid bodies of interest in civil engineering are supported by something as indicated in Fig. 1.1. The supports of the solid body can be regarded as the points where the body meets the external world.

In the case of a bulk mass like a pyramid, the support is obviously the soil beneath the mass. The soil mass however can also be regarded as a solid deformable body since it experiences internal stresses when supporting the pyramid. In this case, you may ask what is the support for that soil mass? With increasing distance from the pyramid's base, within the soil mass the internal stresses continually decrease and eventually are so small that beyond that portion of the soil mass it is not affected by the pyramid. This portion of the overall soil mass that does not deform hence experience internal stress, is the support for the portion of the soil that does experience internal stress.

In the case of frameworks, their individual 1D components (i.e. the beams for plane and space frames, and the ties/struts for plane and space trusses) meet one another at the joints. Another word for joint is "node". Although internal, the joint can be considered to function like an internal support for the members connected to it. In this way supports and joints are similar. More importantly, supports and joints are also similar because the reactions within them are the first quantities the engineer must calculate in order to subsequently calculate the stresses and deformations within the 1D components.

**IF # 7: The reactions at supports and joints are always the first things the engineer must calculate for a framework under applied forces.**

Though supports and joints are similar, the supports have a special duty - they provide the reactions to the external applied forces that are needed to ensure that the solid body is not unstable. An unstable body is one which will move off its supports when external forces are applied.

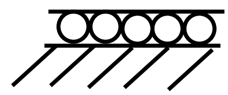
**IF # 8: Supports have the special function of providing the reactions to the external applied forces that are needed to ensure that the solid body will not move off its supports when external forces are applied.**

It is therefore very important for us to know what kind of supports exist, and what reactions they provide.

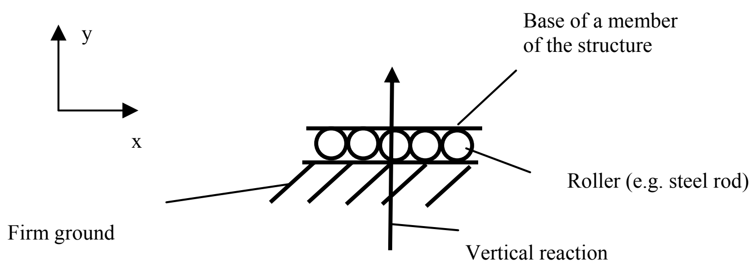
1. Types of Supports in 2D:

In this section we discuss some of the main types of supports for 2D solids.

1.1 The Roller Support



(a) Typical symbol for Roller Support




(b) Interpretation of Roller Support

Fig. 1.11 The Roller Support

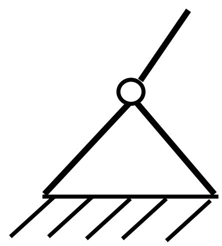
Fig. 1.11 (a) shows the typical symbol for a roller support, and (b) shows the meaning of the symbol. As indicated in (b), the base of the structure's member is represented by the top horizontal line, the rollers by the circles, and the firm ground by the bottom horizontal line and the slanting lines.

The presence of the rollers means that if a force is applied from the base in the x-direction, the support cannot resist this force since the rollers will cause the structure to slide on the ground. However, a vertical force acting downwards (-y-direction) will be resisted by a vertical reaction acting upwards (+y-direction) due to the presence of the firm ground. Of course, if the vertical force from the member was acting upwards, the vertical reaction will act downwards.

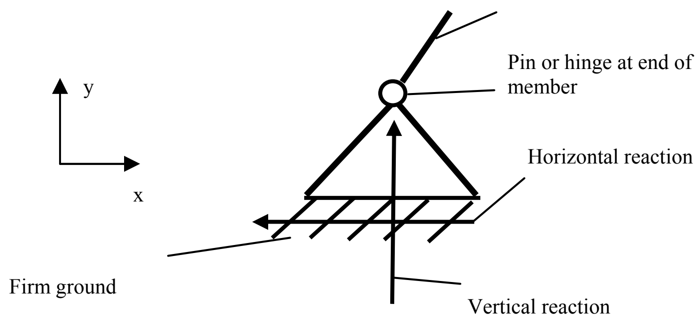
We used rollers to cause the member to be able to slide horizontally, but we can use other devices. Instead of a roller another common device causing the same effect is a smooth frictionless surface.

Therefore, a roller support can provide only one reaction.

1.2 The Pinned Support



(a) Typical symbol for Pinned Support



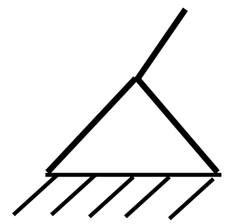

(b) Interpretation of Pinned Support

Fig. 1.12 The Pinned Support

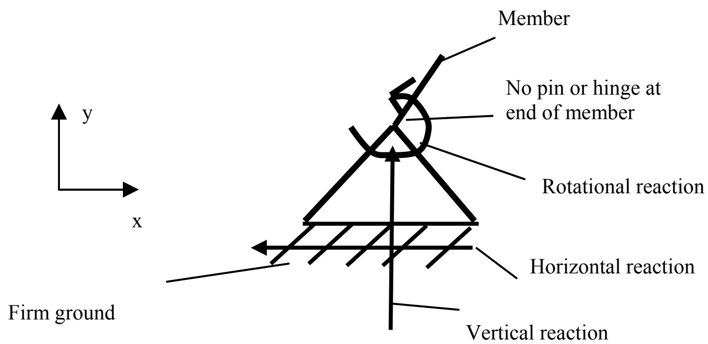
The pinned support is also called a "hinged" support. The key thing is that at the end of the member there is a pin or hinge. This has the effect of allowing the member to rotate at the point where the member meets the support (like a door hinge does). Because of this, the support cannot resist this rotation. However, the support resists sliding in a horizontal direction and can develop a reaction in that direction.

Therefore, a pinned support can provide two reactions (one vertical and one horizontal).

## 1.3 The Fixed Support



(a) Typical symbol for Fixed Support




(b) Interpretation of Fixed Support

Fig. 1.13 The Fixed Support

The fixed support is similar to the pinned support except that at the end of the member, there is no pin to allow the member to rotate. Therefore at the based of the member, the support provides a resistance or reaction to rotation. As for the other types of support, the directions of the reactions depend on the directions of the applied forces.

Therefore, a fixed support can provide three reactions (one vertical, one horizontal, and one rotational).

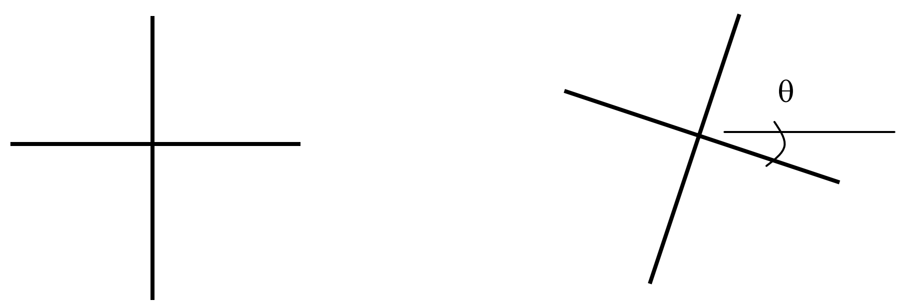
**IF # 9: Roller supports (including frictionless surfaces) provide 1 vertical reaction. Pinned supports provide 1 horizontal and 1 vertical reaction. Fixed supports provide 1 horizontal reaction, 1 vertical reaction, and a rotational reaction.**

## 2. Types of Joints in 2D:

As stated earlier, joints and supports are similar in that they both provide the reactions to the applied forces. However, the important difference with joints is that they can enable forces to be transferred across the joint to the other members that meet at the joint. This is how structures work and the bulk of what we learn in structural mechanics centers on this phenomenon. Once we know the reactions at the ends of the members, it is then very easy to calculate the internal stresses and deformations at any section of the member. We examine this in greater detail in subsequent sections.

### 2.1 The Rigid Joint

A rigid joint is a joint in which all the members meeting at the joint rotate by the same amount when the forces are applied to any of the members.




(a) Before forces applied (a) After forces applied

Fig. 1.14 The Rigid Joint

For example, in Fig. 1.14, 4 members meet at the joint as shown in (a). After forces are applied however (not shown), the arrangement is as shown in (b). Notice that all the members have rotated clockwise by the same amount θ. Hence the joint is a rigid joint. Furthermore, for a rigid joint the angles between the members remain the same before and after the load is applied, even if the joint rotates.

As in the case of the fixed support of section 1.3, a member at a rigid joint (we also say a "rigidly connected" member) will have a rotational reaction where the member meets the joint.

In real structures, concrete frames typically have rigid joints. In steel-framed structures, special construction is required to create a rigid joint.

However, in the case of any beam which is continuous over the supports, then at each support the intersection of the beam and the support acts like a rigid joint hence there are always rotational reactions in the beam at either side of each support. This is shown in Fig. 1.15 below. Note that the leftmost joint is a pinned joint since the beam is not continuous over the support there.

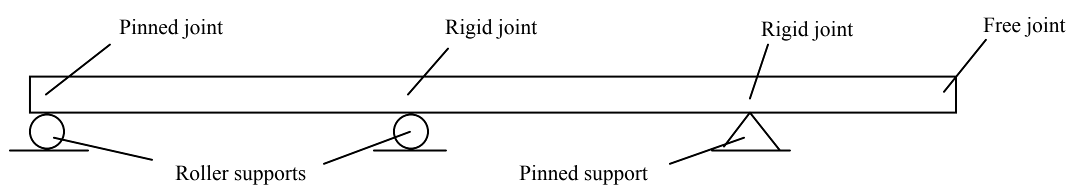
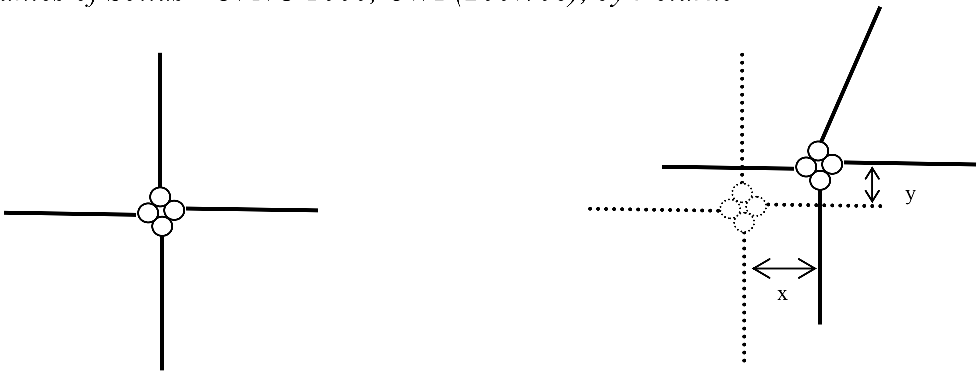



Fig. 1.15 Joints in a Continuous Beam

**IF # 10: A rigid joint is a joint in which all the members meeting at the joint rotate by the same amount when the forces are applied to any of the members.**

## 2.2 <u>The Pinned Joint</u>

A pinned joint is a joint where all the members have pins or hinges at their ends, similar to the pinned support of section 1.2. The pin has the effect of preventing the member from causing the joint to rotate if a force is applied to that member. Hence pinned joints do not rotate, but they can move (i.e. translate) to a point in the plane. Note also that though the joint does not rotate, a member can rotate if a force is applied on the member.




(a) Before forces applied (a) After forces applied

Fig. 1.16 The Pinned Joint

Most planar trusses have pinned joints.

**IF # 11: A pinned joint is a joint where all the members have pins or hinges at their ends, with the effect of preventing the member from causing the joint to rotate if a force is applied to that member.**

### 2.3 <u>The Free Joint</u>

An example of a free joint is shown in Fig. 1.15. Hence a free joint has only one member connected to it (in its plane) and no support below. A free joint has no translational (i.e. horizontal or vertical) or rotational reactions and therefore moves freely without any restraint. They are quite practical for example in providing overhanging floors or eaves to prevent the ingress of rain.

**IF # 12: A free joint has only one member connected to it (in its plane) and no support below.**

## 2.0 FORCES AND STATIC EQUILIBRIUM

In section 1.0 we examined what we mean by a solid and said that there are basically 2 types – frameworks and the bulk mass. We also noted that to simplify the calculations frameworks are idealized as collections of subsets of the frameworks called frames and trusses, then further into beams and ties/struts respectively.

For the remainder of this course (with the exception of the last section) we focus on continuous beams and trusses as our solids of interest.

Noting that we stated our activity in "Mechanics of Solids" as the calculating of the stresses and deformations within solids when under the action of external applied forces or loads, in this section we examine the forces on the structure and the conditions for the balance of these forces, also called the equilibrium of the structure.

### a. "The forces are external; the stresses/deformations internal"

The title of this section - "The forces are external; the stresses/deformations internal" is meant to immediately draw attention to the essential features of any structure in terms of the forces acting on it.

What must be understood is that the applied forces on the structure induce reactions at the supports and joints which also act on the structure as if they are applied forces. Hence both the applied forces and the reactions at the supports and joints are the external forces that together generate stresses within the members of the structure.

We may say that for any structure there are 3 kinds of balance of forces occurring. Forces that balance each other are said to be in **equilibrium**.

First, a condition of "external equilibrium" is established when the applied forces on all the members of the structure are exactly balanced by the reactions at the supports only.

Second, a condition of "joint equilibrium" is established at each joint within the structure.

Third, a condition of "internal equilibrium" is established when for each member the forces (stresses) at each section within the member exactly balance both the applied force on the member and the reactions at the joints at the ends of the member.

These 3 kinds of balance of forces occur simultaneously on and in the structure and the overall structure is said to be in static equilibrium since it only deforms internally and not as a whole, therefore remaining at rest.

The deformations of the structure occur because the material of which the solid or the members of the structure are composed deforms under the action of the internal stresses.

We have presented the 3 kinds of equilibrium in the order indicated because this is the order in which we perform the calculations.

**IF #13 Any solid or structure is under a set of external loads comprised of <u>both</u> the applied forces, and the reactions to those forces at the supports and the joints. The reactions at the supports exactly balance the total applied forces.**

**IF #14 We may say that there are 3 kinds of balance of forces occurring in a structure – the balance of the total applied forces with the reactions at the supports; the balance of the forces occurring at each joint, and the balance of the internal forces at any section of a member with the forces acting on the member and at the ends of the member.**

b. <u>Forces – Resultants and Equilibrants:</u>

In the last section we noted 3 types of equilibrium of forces on a structure. We also noted that equilibrium occurs when forces balance each other. Therefore it is very important for us to understand the nature and properties of these forces.

Any set of forces can be combined to form one force called the **resultant.** For a body to be in static equilibrium (i.e. at rest or uniform motion) the resultant of ***<u>all</u>*** forces acting on the body must equal exactly zero. This means that if one set of forces acting on the body has a non-zero resultant there must also be another set of forces acting on the body but with a resultant equal and opposite to the resultant of the first set of forces, for the body to be in static equilibrium. This second set of forces required to balance the first set, is called the **equilibrant.**

For example, with respect to the first type of equilibrium we discussed earlier, the external applied forces have a resultant but the reactions at the supports provide the equilibrant for a structure in static equilibrium.

In this section we examine the balancing act between resultants and equilibrants for 2 classes of force systems – concurrent and non-concurrent. Furthermore, we limit our presentation to forces that are all in the same plane. Such a force system is said to be **coplanar.**

The practical applications of coplanar forces are the mechanics of 2-dimensional frameworks. In this book, we examine the coplanar forces on continuous beams and trusses only.

1. <u>Equilibrium of Concurrent Forces</u>

In section 2a it was said that in a structure the forces at the joints are in equilibrium with each other (the second kind of equilibrium). If the structure is a planar truss, then at a typical joint the **lines of action** of the forces all meet at a common point. Such forces are said to be **concurrent** (they all meet at the same point), and **coplanar** (they all act in the same plane – the plane of the truss).

Calculations for coplanar forces can be done using 2 approaches – graphical or mathematical (algebra and trigonometry).

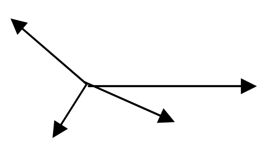
This is because, as we know from elementary physics, a force is completely defined by its magnitude and direction and is therefore a vector quantity. Hence when vectors are 2-dimensional we can draw them on paper, resulting in the graphical approach. But, vectors can be treated mathematically as they obey the additive, multiplicative and transformation laws of vectors, hence the mathematical approach.

In the graphical approach the magnitude of a force is represented by the length of a line and the direction is represented by the angle of the line from a convenient axis, and an arrow (also called the "sense" of the vector).

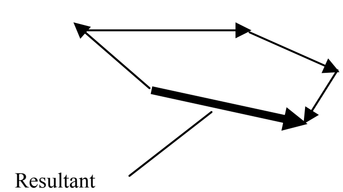
<u>Resultants by Vector Addition:</u>

The resultant of a set of concurrent coplanar forces is determined by arranging the forces as vectors in such a way that the arrows of all the force vectors follow each other in turn. The resultant is then found simply as the vector connecting the start and end.

Example 2.1: Find the resultant and equilibrant of the following concurrent forces:



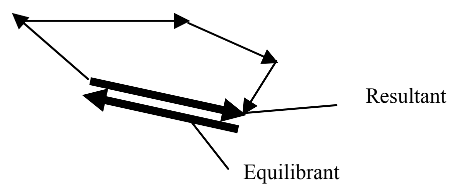
Rearranging so that the arrows follow each other (called the head-to-tail rule) in turn we get.



This is called a **polygon of forces**.

A key thing to remember about forces is that it is the net effect of the forces that are important and you can arrive at this by combining forces in any manner. This is the same as saying that you can get from the start to the end by any route.

The equilibrant of the forces is simply the force equal and opposite to the resultant. Hence the equilibrant must pass through the same point as the resultant. The resultant and equilibrant therefore are a system of concurrent forces.




If this resultant and equilibrant were applied to a body it would remain at rest since they exactly balance each other. For an actual structure, you do not have to apply the equilibrant – it automatically arises (if the supports allow) to keep the body in equilibrium. The resultant in this case therefore represents the net effect the external forces applied to or **imposed** on the structure.

Notice in the above figure that the equilibrant and the other forces form a **closed polygon** with the arrows at the sides of the polygon following each other. Hence we get the important fact that concurrent forces in equilibrium always result in a closed polygon of forces.

**IF #15 Concurrent forces in equilibrium always result in a closed polygon of forces.**

Special Case of 3 Coplanar Forces in Equilibrium:

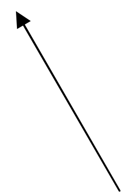
If 3 coplanar forces are in equilibrium they must be either (i) parallel forces, or (ii) concurrent forces. The special application of (ii) is that if we know the magnitude of only 1 of the forces but the lines of action of the other 2, we can easily draw a closed polygon of forces and find the magnitude of the other 2 forces.

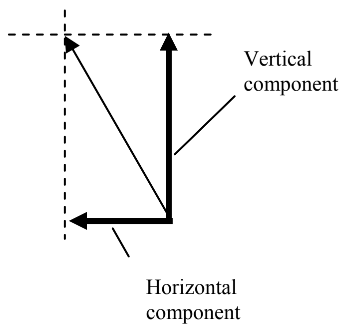
**IF #16 3 coplanar forces in equilibrium must be either parallel or concurrent forces.**

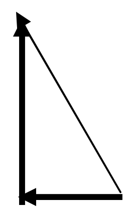
Components of a Force:

It is sometimes very convenient when doing calculations to break up a force into several forces. The procedure of breaking up a force into other forces is called **resolution**, and the resulting forces are called **components** of the force**.** If these forces are 2 forces *at right angles to each other*, they are called the **rectangular components** of the force and they are in the horizontal and vertical directions.

Example 2.2: Find the components of the following force.





Head-to-tail confirmation that the sum of the components equals the force

If we examine a force and its component closely, we notice that a force can never have a component at right angles to the force. Put another way - a force never has any effect at right-angles to its line of action. This is one of the most important facts in all of engineering mechanics and is called orthogonality.

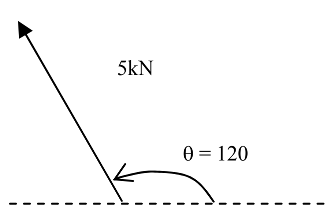
**IF #17 A force can never have a component at right angles to the force.**

<u>Resultants by Trigonometry:</u>

The components of a force are given by the sine and cosine of the force. If we measure the direction of the force by an angle θ which is +ve anticlockwise from the horizontal axis, then for a resultant R at angle θ,

Horizontal component of R = $H_R$ = $R\cos\theta$
Vertical component of R = $V_R$ = $R\sin\theta$.

Hence for the following force of magnitude 5kN and θ = 120 deg



$H_R$ = 5cos120 = -2.5 kN
$V_R$ = 5sin120 = 4.33 kN

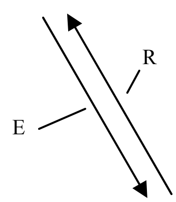
Notice that when we measure θ as +ve anticlockwise from the horizontal axis, then a –ve $H_R$ means that its direction is to the left and vice versa, and a –ve $V_R$ means that its direction is downwards and vice versa.

Also,

$R = \sqrt{( H_R^2 + V_R^2)}$ and $\tan\theta = (V_R / H_R)$.

Conditions of Static Equilibrium for Concurrent Forces:

As for the graphical approach, for static equilibrium the resultant, R, but be balanced by the equilibrant, E (equal and opposite to the resultant).




The only difference between R and E is that for E, θ = -60 deg.

If we add R and E in terms of their components we get:

Horizontal components of R+E = $H_R + H_E$ = 5cos120 + 5cos(-60) = -2.5+2.5 = 0
Vertical components of R+E = $V_R + V_E$ = 5sin120 + 5sin(-60) = 4.33+(-4.33) = 0

Since the resultant can be resolved into any number of forces, and likewise for the equilibrant, we have therefore demonstrated that *for any set of coplanar concurrent forces in static equilibrium*, the sum of the horizontal components of all the forces (i.e. ∑H) is zero and the sum of the vertical components of all the forces (i.e. ∑V) is zero.

**IF #18 For a system of coplanar concurrent forces in equilibrium - ∑H=∑V=0**

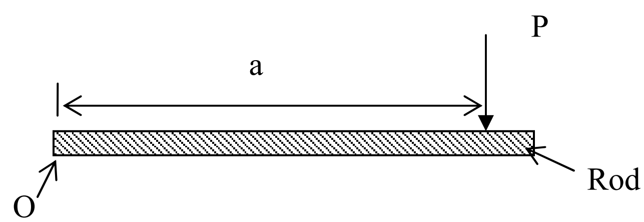
## 2. Equilibrium of Non-Concurrent Forces

The conditions of static equilibrium of coplanar non-concurrent forces are the same as for concurrent forces except for one important difference - the need to consider the tendency for the non-concurrent forces to cause the body they act on to rotate as a whole. Therefore, we need to introduce 2 new concepts - the **position** of a force, and the **moment** of a force. For the following we limit our discussion to parallel non-concurrent forces, since these are of greater practical interest, especially for beams.

### Position and Moment of a Force:

The position of a force is the location of the point of application of the force on the body, and measured from a convenient origin.

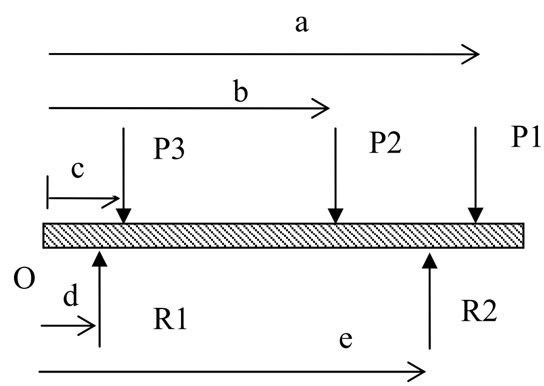
The moment of a force is a measure of the force's tendency to cause rotation about a point. It is defined as the product of the magnitude of the force, and the perpendicular distance from the line of action of the force from the point.




Consider the rod above. The force P is at position "a" from origin O, and the moment of P relative to O is Pa. The typical way of saying this that the moment of P **about** O is Pa. "a" is called the **lever arm**.

The moment of P about O causes the rod to rotate clockwise about O. Clearly, this system is not in equilibrium since it results in a rotation of the rod, whereas equilibrium means at rest.

Now consider several non-concurrent forces acting on the rod.




In this diagram P1, P2, and P3 are applied loads, and R1 and R2 are reactions to those loads. Remember, from the standpoint of the rod, all the forces (i.e. P1, P2, P3, R1 and R2) are external forces.

<u>For this system of external forces to be in static equilibrium</u>, the rod must (i) not move up or down, and (ii) it must not rotate.

For (i) this means, P1 + P2 + P3 = R1 + R2
And for (ii) this means, sum of clockwise moments of all forces = sum of anti-clockwise moments of all forces

Condition (i) is the same as for concurrent forces in that $\sum V=0$, or P1 + P2 + P3 - R1 - R2 = 0 (if we take down as positive).

But for (ii) we have a new condition that is unique to non-concurrent forces - the sum of the moments of all forces must equal zero.

At this point the question arises "the sum of moments about which point?" And the answer is, the sum of moments about any point in the plane of the forces. If we choose point O, then we get,

Sum of clockwise moments = (P1xa)+(P2xb)+(P3xc)
Sum of anti-clockwise moments = (R1xd)+(R2xe)

In other words, (P1xa)+(P2xb)+(P3xc)-(R1xd)-(R2xe) = 0 (if we take a clockwise moment as positive).

The reason why it does not matter where we take moments from is that the lever arms of all the forces will change proportionately, so we will always get back the same moment equilibrium equation.

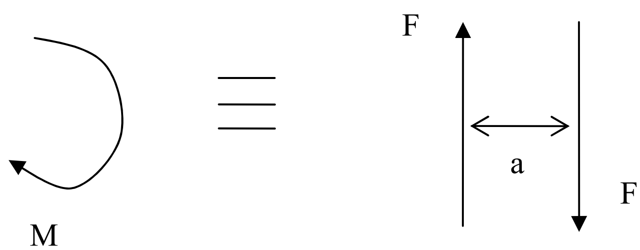
Hence the new condition of equilibrium for non-concurrent forces is: $\sum M_c = 0$, where c is any point in the plane of the forces.

**IF #19 For a system of coplanar non-concurrent forces in equilibrium -** $\mathbf{\sum H = \sum V = \sum M_c = 0}$**.**

A Moment as a Couple:

In the same way that it is sometimes convenient in calculations to break up a force into components, it is sometimes convenient to break up a moment into a **couple**.

A couple is a pair of equal but opposite forces separated by a distance or lever arm, a. For example, the following clockwise moment M can be considered equivalent to the couple Fa.



c. Statical Determinacy and Geometric Instability:

In section 2a entitled "The forces are external; the stresses/deformations internal" we identified 3 types of equilibrium occurring in a typical structure - (a) the equilibrium between the applied loads, (b) the equilibrium of the joints, and (c) the equilibrium of the forces (stresses) at any section of a member of the structure. We consider only (c) to be internal since it is within the material of which the member is composed.

Let us look at this for the case of the 2 main types of solids we are concerned with throughout our presentation - the plane truss and the continuous beam.

1. Determinacy of Plane Trusses

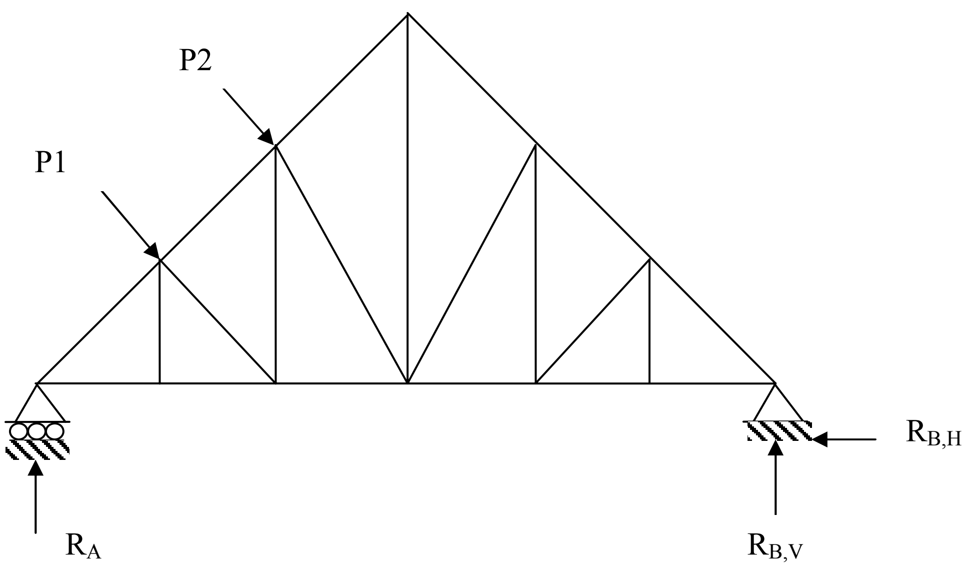



Fig. 2a Example of Equilibrium Type (a) - Applied loads P1, P2 balanced by reactions $R_A$, $R_{B,V}$ and $R_{B,H}$

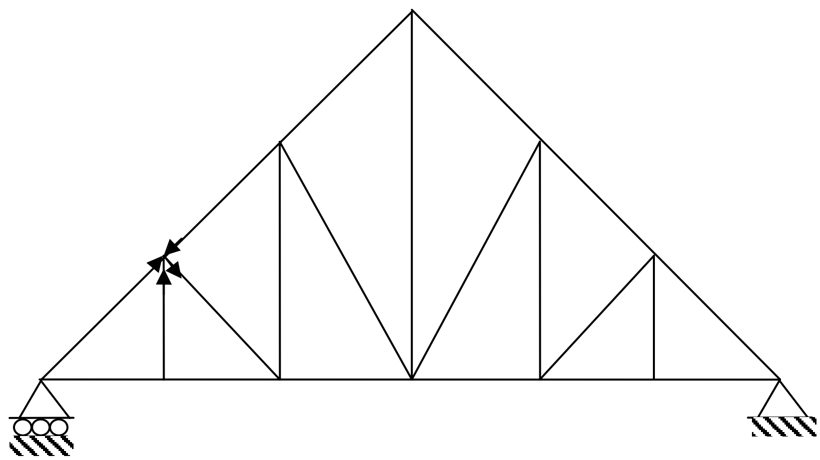


Fig. 2b Example of Equilibrium Type (b) -
The forces meeting at the joint must balance each other (i.e. have a zero resultant).

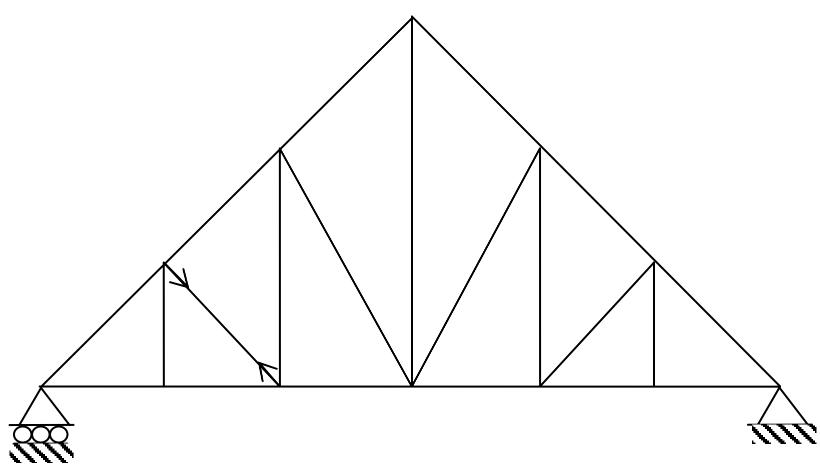


Fig. 2c Example of Equilibrium Type (c) -
The forces at any sections of a member must balance each other (i.e. have a zero resultant).

Figs. 2a, b, c are self-explanatory (in b and c only the forces at 1 joint and 1 section are shown for simplicity). However, it must be understood that all the forces shown exist simultaneously, but only forces P1 and P2 are the applied forces which are known.

The other forces, that is the reactions $R_A$, $R_{B,V}$ and $R_{B,H}$ , and the forces at the joints, must be calculated and is our main purpose. These are our "**unknowns**".

How many "unknowns" are there in general for a planar trusses? The total number of unknowns are (1) the sum of the reactions, plus (2) the sum of the forces at all joints.

Let us call (1) as "r". Note that we treat the rectangular components of the reactions as individual reactions so in our example r = 3. (But actually there is only 1 force at B which is at an angle to the horizontal and which we get by vector addition of its components).

Now if we examine Fig. 2b we notice that at a joint the number of unknown forces is the same as the number of members at the joint. So the total number of unknown forces at all the joints of a planar truss is the same as the total number of members of the truss. Let us call this as "m".

Hence the total number of unknowns is r+m, and this is what we must determine. Since we are using mathematical procedures to calculate for these unknowns, we must have enough information to form r+m simultaneous equations.

A **statically determinate** structure (regardless of the type of structure) is one where we can get the information we need by using the statics equations only. Hence "statically determinate" means "determined by using statics only".

At any joint or support of a truss all the forces are concurrent. For a truss, we know from IF#18, which is the condition of static equilibrium for concurrent forces, that $\sum H=0$ and $\sum V=0$. This is 2 equations. Hence the total number of statics equations for a truss is 2j, where j is the total number of joints including the supports.

Hence we can now state the condition of **statical determinacy** for a truss as:

m+r = 2j. Note that the condition of statical determinacy is independent of the applied loads on the structure. Let us check our truss to see if it is statically determinate.

m = 21 r = 3 j = 12

Hence m+r = 21+3 = 24 and 2j = 2x12 = 24, therefore the structure is statically determinate.

If m+r>2j, this means that there are more unknowns than the information we have from statics for us to get the number of equations we need to solve for the unknowns. Such a structure is said to be **statically indeterminate**, or **redundant**, or **hyperstatic**.

For redundant structures we get the additional equations required by considering the geometry of deformation of the structure, resulting in more complex methods of calculation. We are introduced to such structures in Level 2 via the course "Structural Mechanics". In our Mechanics of Solids course however, we study only statically determinate structures.

If m+r<2j the structure is called a **mechanism** and will collapse if any loads are applied to it because there are either an insufficient number of members, or the supports do not provide an appropriate number and type of reactions.

This suggests that a statically determinate structure is a stable structure but this is not necessarily so as we will examine in the last section of this chapter.

**IF #20 A statically determinate structure is one where the unknown forces at the supports and joints can be determined by using the equations of static equilibrium only. For a statically indeterminate structure, also called a redundant or hyperstatic structure, the geometry of deformation of the structure must be considered in order to obtain the remaining equations for solution.**

**IF #21 For a planar truss the condition of statical determinacy is that m+r = 2j where m is the number of members, r is the number of reactions (after converting to rectangular components), and j is the number of joints including the supports.**

**IF #22 For a statically indeterminate truss m+r > 2j.**

**IF #23 If m+r < 2j, the structure is called a mechanism and will collapse under applied loads.**

## 2. Determinacy of Beams

The determinacy of continuous beams is conceptually similar to that of trusses but with notable differences. Let us examine a typical continuous beam as was done in the last section for planar trusses, starting with the 3 types of equilibrium in a structure.

Consider a simple 2-span continuous beam. Note that though a continuous beam is physically really 1 beam, we usually speak of a portion of the beam between supports as if it were a separate beam connected to an adjacent beam via the joint over the support. (Refer to Chapter 1 Section b1, and Fig. 1.15). Hence our 2-span continuous beam is considered as 2 beams connected via a rigid joint over the central support.

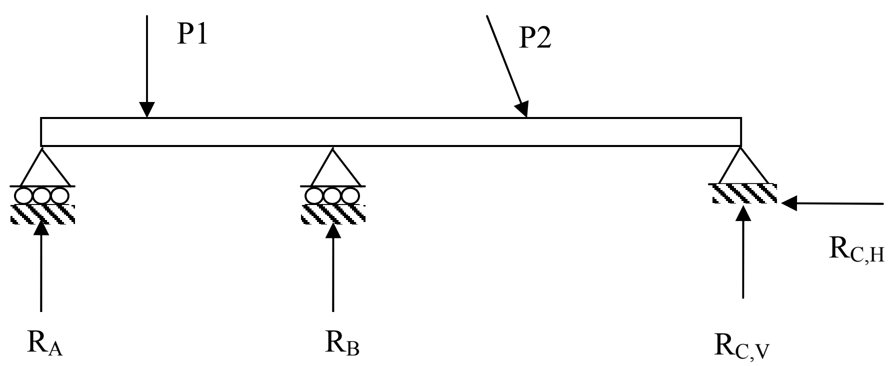



Fig. 3a Example of Equilibrium Type (a) - Applied loads P1, P2 balanced by reactions $R_A$, $R_B$, $R_{C,V}$ and $R_{C,H}$

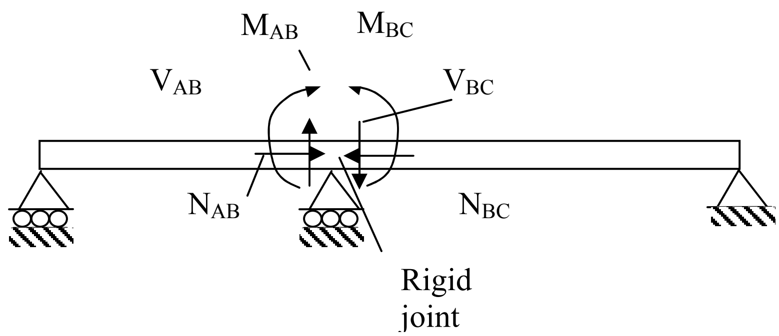



Fig. 3b Example of Equilibrium Type (b) - At the joint, the Ms must balance each other (i.e. have a zero resultant), the Vs must balance each other, as well as the Ns

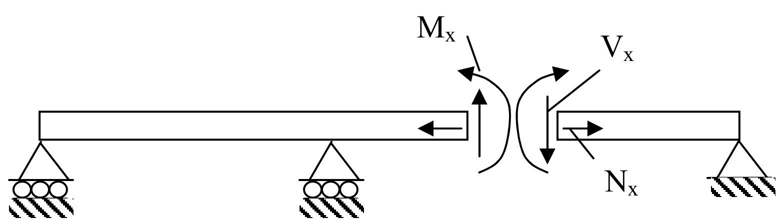



Fig. 3c Example of Equilibrium Type (c) - At a section, the Ms, Vs and Ns, must balance the applied loads and reactions (i.e. have zero resultants).

Figs. 3a, b, c are self-explanatory (in b and c only the forces at 1 joint and 1 section are shown for simplicity). One difference when compared with a truss, is that it may not be immediately obvious that the zone of the beam immediately above a support is a joint (Fig. 3b).

However, it must be understood that all the forces shown exist simultaneously, but only forces P1 and P2 are the applied forces which are known. The other forces, that is the reactions $R_A$, $R_{B,V}$ and $R_{B,H}$ , and the forces at the joints, must be calculated and is our main purpose. These are our "**unknowns"**.

How many "unknowns" are there in general for a continuous beam? The total number of unknowns are (1) the sum of the reactions, plus (2) the sum of the forces at all joints.

Using the same notation as for the truss discussed in the previous section, "r" is the number of reactions, so in our example r = 4. Likewise, "m" is the number of beams, so in our example, m = 2.

Now if we examine Fig. 3b we notice that at the end of a beam there are 3 unknown forces - an M, a V and a P. Since these are 3 forces, the number of unknown forces at the joints is the same as 3 times the number of members at the joint. So the total number of unknown forces at all the joints of a continuous beam is the same as 3 times the total number of beams. Let us call this as "3m".

Hence the total number of unknowns is r+2m, and this is what we must determine. Since we are using mathematical procedures to calculate for these unknowns, <u>we must have enough information to form r+3m simultaneous equations</u>.

At any joint of a continuous beam, since a moment is equivalent to a couple (see section b2), the forces are non-concurrent. Hence at a joint in a continuous beam, we know from IF#19, which is the condition of static equilibrium for non-concurrent forces, that $\sum H=0$, $\sum V=0$, $\sum M_C=0$. This is 3 equations. Hence the total number of statics equations for a continuous beam is 3j, where j is the total number of joints <u>including the supports</u>.

Therefore we can now state the condition of statical determinacy for a continuous beam as:

3m+r = 3j. Note that the condition of statical determinacy is independent of the applied loads on the structure. Let us check our continuous to see if it is statically determinate.

m = 2 r = 4 j = 3

Hence 3m+r = 3x2+4 = 10 and 3j = 3x3 = 9, therefore the structure is statically indeterminate. The total number of unknowns minus the total number of statics equations is called the **degree of indeterminacy**. Hence our continuous beam has 10-9 = 1 degree of indeterminacy.

Special Case of Condition of Determinacy for Continuous Beams:

The condition of determinacy for continuous beams of 3m+r = 3j is the general case in that horizontal forces and forces along the length of the member are considered. This arises when there is an applied load that has a horizontal component, such as load P2 of our continuous beam.

But this is an impractical situation for a typical beam and it is much more common for continuous beams to carry only vertical applied loads. When this is the case, there is no force in the beams along their length, and no horizontal reaction as well. Hence the condition of determinacy becomes: 2m+r = 2j. Applying this special case to our continuous beam we now get 2m+r = 2x2+3 = 7, and 2j = 2x3 = 6, so the degree of indeterminacy is 7-6 = 1, as we expect.

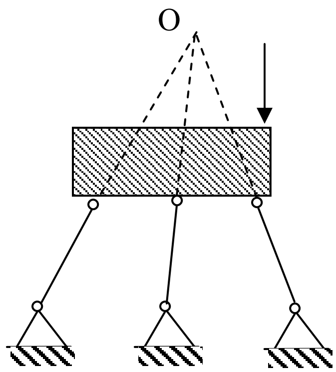
Calculations for a continuous beam are therefore beyond the scope of our presentation of the Mechanics of Solids since they are statically indeterminate. However, we do examine single-span beams in some detail in Chapter 4, as they are statically determinate.

**IF #24 For a continuous beam the condition of statical determinacy is that 3m+r = 3j if there are applied loads with horizontal components, but 2m+r = 2j if the applied loads are vertical only, where m is the number of members, r is the number of reactions (after converting to rectangular components), and j is the number of joints including the supports.**

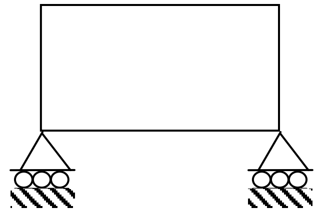
3. Geometric Instability

It was stated earlier that a statically determinate structure is not necessarily stable. We are referring here to geometric stability in which case if a structure is geometrically unstable, the entire structure as a whole will move if a load is applied in a certain direction. When a structure moves as a whole it is called a **rigid body motion**.

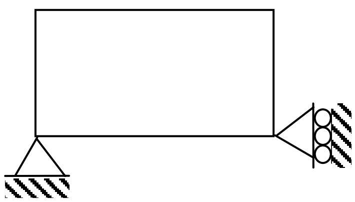
A planar structure will be geometrically stable if for any direction of a load applied to the structure its supports can provide (1) a vertical reaction, (2) a horizontal reaction, and (3) a rotational reaction. The latter cannot happen if the lines of action of all its support reactions pass through one common point. This is because an applied force on the structure will have a non-zero moment about that point, but that moment cannot be balanced by any of the reactions since their lever arms relative to that point are zero. The diagrams below show examples of geometrically unstable structures.



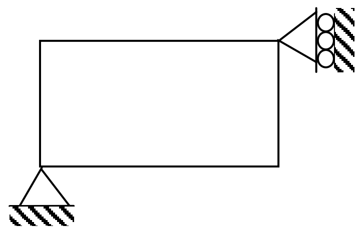

The supports cannot resist the moment of the applied load about O.



The supports cannot resist a horizontal load.

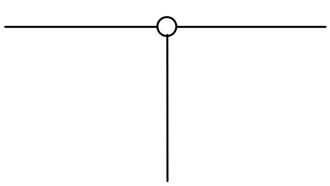


Unstable



Stable

A structure can also be internally unstable. For example in a statically determinate pin-jointed structure such as a truss, if a joint has members that are all vertical or horizontal then that joint will move as a rigid body. This is because, as we remember from IF #17, vertical members will not be able to provide a horizontal reaction, and horizontal members will not be able to provide a vertical reaction. An example of this is shown below.



**IF #25 A structure may be statically determinate yet be unstable. The geometrical arrangement of the supports and the members of a structure must be carefully considered to assure a stable structure.**

## 3.0 STATICALLY DETERMINATE TRUSSES – "REACTIONS THEN JOINTS, THEN SECTIONS"

In this chapter we apply what we learned about the equilibrium of concurrent forces to determine the forces in the members in a statically determinate truss.

We use the graphical representation of forces to develop the graphical method of solution in which we determine the forces by using the facts about the equilibrium of a joint. The graphical methods require drawing the forces to scale so we shall attempt problems during our Coursework sessions.

We then present other solution methods based on mathematical calculation. We examine the equilibrium of the joints again but this time we make use of $\sum H=\sum V=0$ to calculate for the unknown forces. Then we present the "Method of Sections", then lastly, the "Method of Tension Coefficients".

### a. Finding the Internal Forces (in Planar Trusses) by Joint Equilibrium:

3. The Graphical Method:

Recall IF #15 that concurrent forces in equilibrium give rise to a closed force polygon when the forces are represented graphically. Recall also IF# 16 that for any 3 forces in equilibrium they must all meet at a common point therefore if we know the magnitude of only one of them but the directions of all of them, we can draw a triangle of forces and get the magnitudes of the other 2 forces.

Regardless of the (statically determinate) truss we are solving for, the steps of solution are always:-

Step 1: Label the forces in accordance with Bow's Notation.

Step 2. Based on IF #15 determine the 2 reactions.

Step 3. Starting at the pinned support (not the roller support), draw the triangle of forces comprising the reaction, and the forces in each of the 2 members.

Step 4. Considering the forces already calculated, go to the next joint with only 2 unknown forces, and based on IF #15, draw the closed force polygon hence determine the magnitude of the 2 forces.

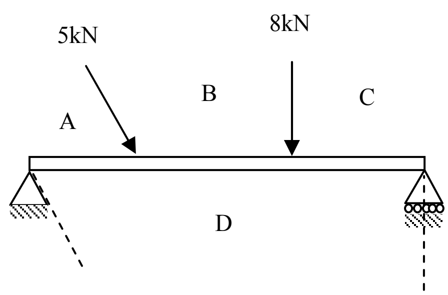
Step 5. Repeat step 4 until all the forces are determined. In the end you have a set of closed polygons connected together.

### 3.1 Bow's Notation and the Force Polygon for the External Forces

In step 1 we said that we must label the forces in accordance with Bow's Notation. Bow's Notation is simply putting letters and numbers in the spaces between the lines of action of all the forces of a structure in equilibrium, regardless of the type of (statically determinate) system. The strength of Bow's Notation however, is that it is used with the "pole" and "rays" to enable a systematic way of drawing a closed force polygon.

Consider the following example. We do not know the magnitudes of the reactions, but we know the line of action of the right reaction since being at a roller support, it must be vertical. It is customary to use capital letters (A, B,…) for external forces, numbers (1,2,…) for member forces, common letters for the "rays", and "O" for the "pole".

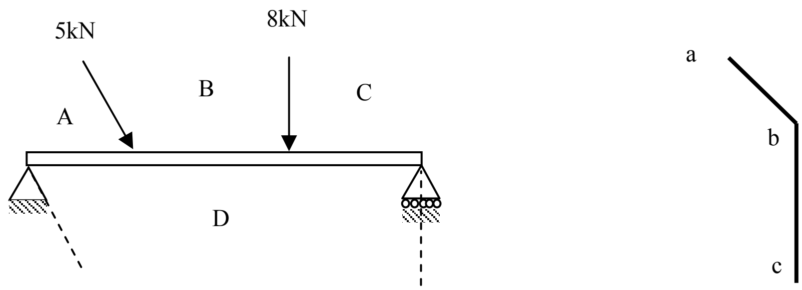
Label the spaces (note that we only have external forces in this case). When using Bow's Notation to determine reactions we must always proceed towards the reaction with the known line of action. Therefore we must go clockwise in this case.



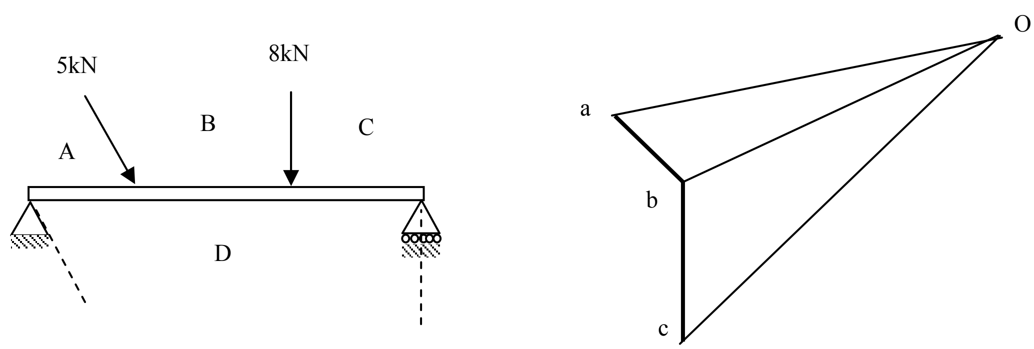

The left reaction is then DA, and AB is the 5kN applied load, BC the 8kN applied load, and CD, the right reaction. Since we are going clockwise, we must remain consistent through the problem and always go clockwise.

**IF #26 When using Bow's Notation to determine reactions, we must always start at the support without the known line of action, and proceed toward the reaction with the known line of action.**

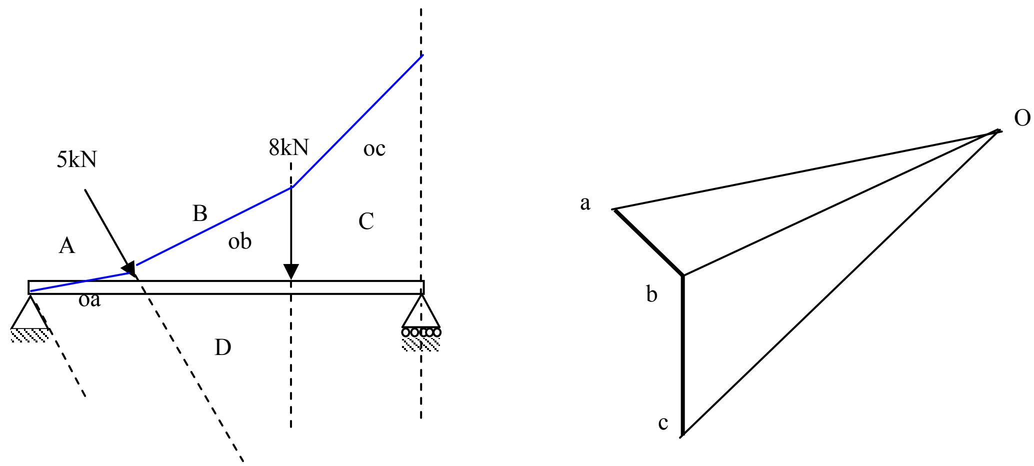
Choose a region on the paper and begin the force polygon to scale. It is typical to use common letters in the force diagram.



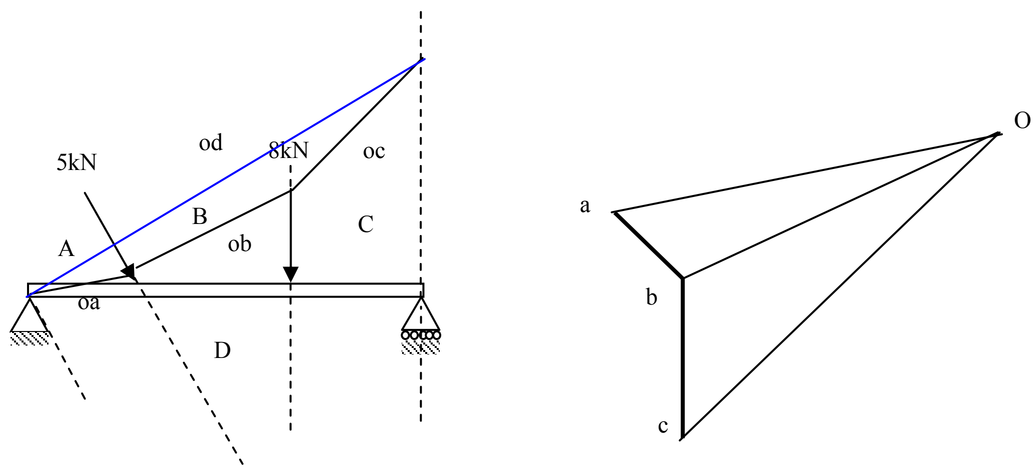

Notice that the succession of the letters in the force diagram gives the sense of the force so "bc" is downwards. Next, choose a point on the paper and label it as "O" - the Pole. Then draw lines connecting the vertices of the force diagram to the pole. These are the rays" - oa, ob, etc.



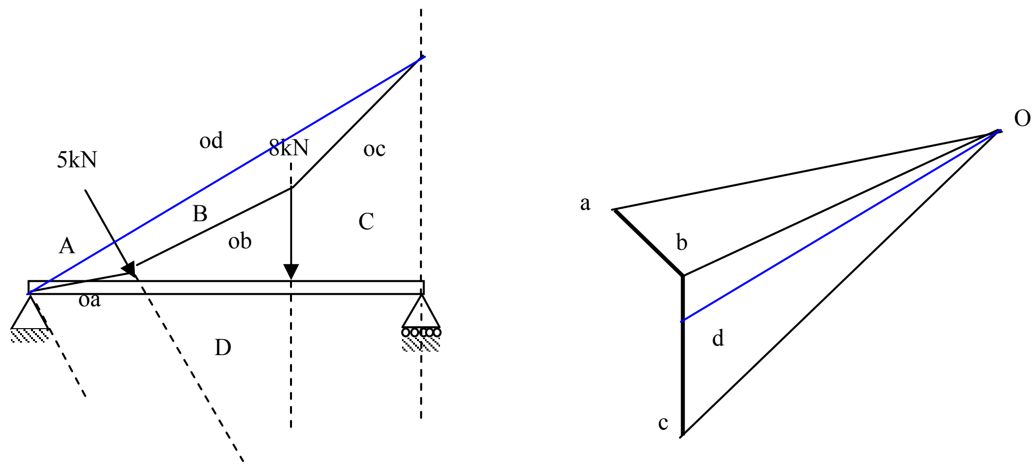

Next, from IF #26 starting at the left support (the one without a known line of action of the reaction), draw lines on the left diagram, called the **space diagram**, parallel to the rays but cutting the line of action of the next force going clockwise. Hence a line parallel to oa must cut the line of action of AB, ob must cut BC, and oc must cut CD. Use 2 set squares to transfer the lines.



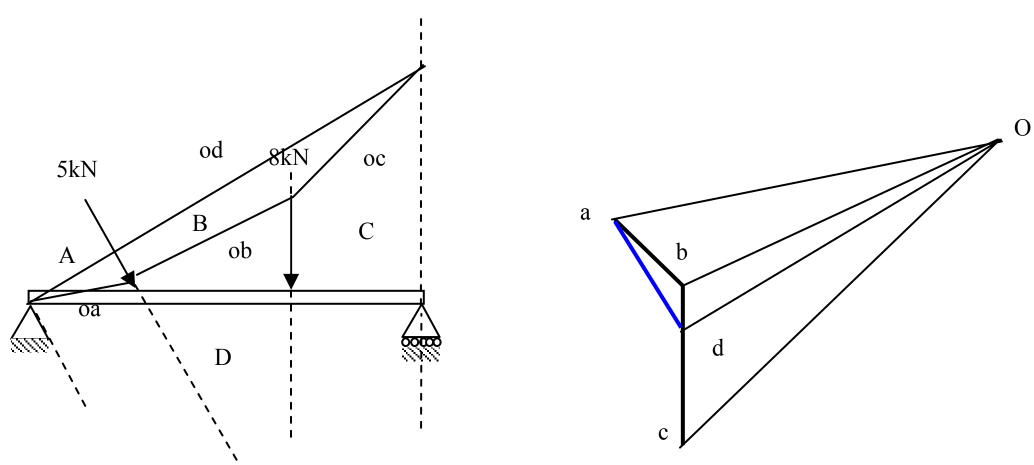

The next step is critical - on the space diagram, close the polygon by drawing a line which will then be od. This is not a force polygon and is called the **link polygon**.




Next, in the "force polygon and rays" diagram, draw a line parallel to od, starting from "O" but cutting a vertical line from c on the force polygon. This gives us force CD. Clearly, CD - the right reaction, is upwards as we expect.




Now in the final step, we get the left reaction DA, simply by closing the force polygon (i.e. by drawing a line from d to the start of the force polygon, a). Remember that this is because for a system of forces in equilibrium, the forces form a closed polygon when drawn. The reactions are in equilibrium with the applied loads, with both being the external loads on the system.

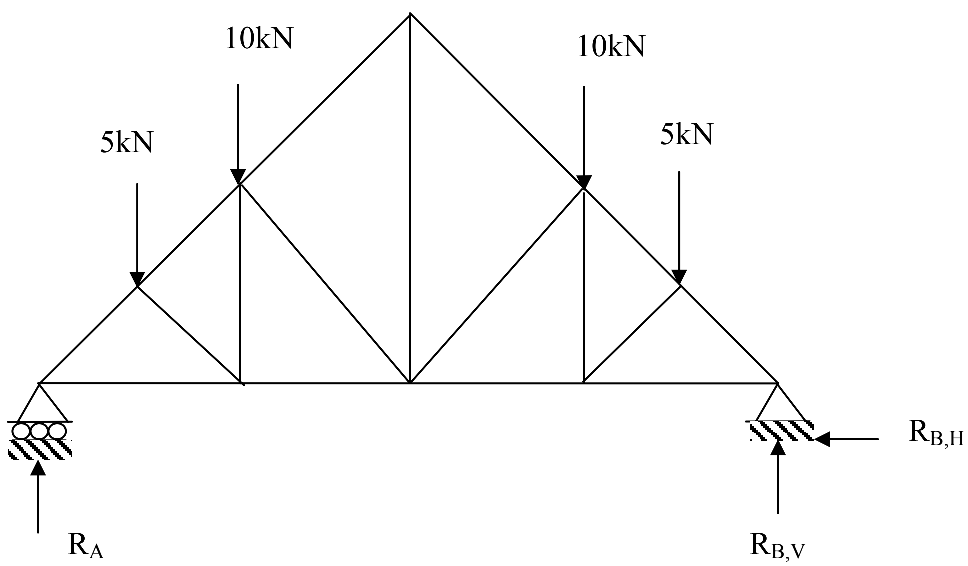


### 3.2 The Force-Polygons at the Joints

Returning to the steps of section 1 for the graphical determination of forces in statically determinate trusses, in the last example we showed how to perform steps 1 and 2, which apply to statically determinate beams or trusses.

To continue the steps for trusses, these are accomplished simply by using the force polygon and adding other closed polygons for each joint. For trusses, unlike in our previous example, when we label the paces between the lines of action of forces, we will need to label the spaces between the truss members as well.

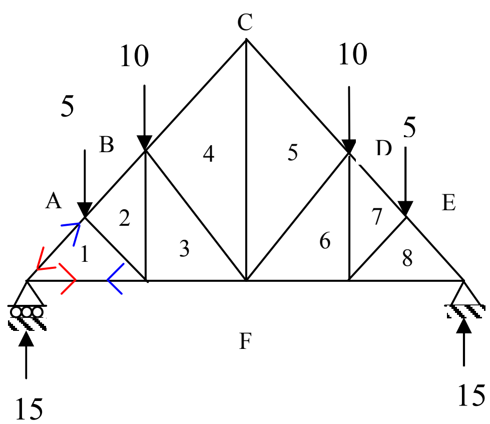
Consider the following example which is typical of all trusses. In order to focus on steps 3 to 5, we choose an example where the reactions are obvious.

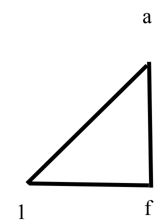



The truss is statically determinate (m+r = 17+3=20; 2j = 2x10=20) and as the structure and applied loads are symmetrical, $R_A = R_B$ = 15kN (i.e. $R_{B,H}$ = 0).

Apply Bow's Notation starting from the left support and going clockwise (it does not matter if we start here since we already know the reactions, but we must go clockwise for all joints). Then, knowing the lines of action of the forces in the members, but both the line of action and magnitude of the reaction at the left support (FA), draw the closed polygon of forces, to scale, for the forces meeting at the left support (i.e. IF #16).

Since we need the lines of action of the forces, draw the space diagram to scale so that we can use 2 set squares to transfer a line to the force diagram. Hence for the left support, we get the closed force polygon shown. This tells us the magnitude of the forces in the members meeting there, and whether the member is under a tensile or compressive force.



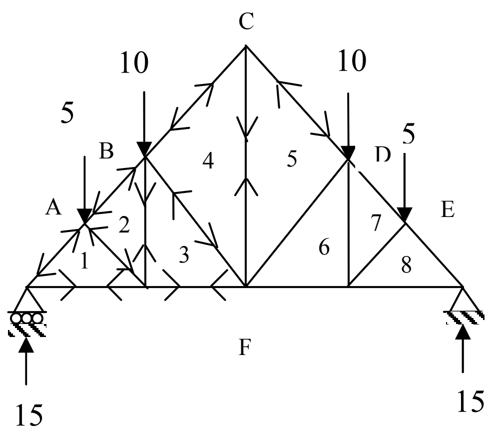

Since the forces are in equilibrium they must follow each other in turn so we must go from "f" to "a" to "1". But a1 (i.e. force "a" to "1"), is the force in the diagonal member so we know that this forces pushes on the joint. Similarly, the force in the horizontal member, 1f, pulls on the joint. These are the red arrows. Since all sections in a member must also be in equilibrium, we know the force at the other end of the member but the arrow is reversed. These are the blue arrows. In the diagonal member the arrows point away from each other so the member is in **compression**. In the horizontal member the arrows point toward each other so the member is in **tension**.

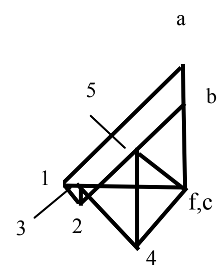
**IF #27 To know whether the force in a truss member is compressive or tensile, if the arrows point away from each other the member is in compression, but if arrows point toward each other, then the member is in tension.**

We must now choose the next joint to draw its force polygon. In this case, we must go to the joint with the 5kN applied load. This is because this joint has only 2 unknown force magnitudes. We must always use this rule. Remember also to use the information about previously calculated forces. For example, at the joint with the 5kN, there are 4 forces - 3 from the members meeting there plus the 5kN load. But we know one of them from the calculation for the previous joint (the blue arrow), so there are only 2 unknown forcs at that joint.

**IF #28 In calculating the forces in truss members, to choose the next joint to work on, it must be a joint with no more than 2 unknown force magnitudes.**

We proceed like this for the remaining joints, remembering to go clockwise around each joint. The result is shown below for little more then one half of the truss because as the truss is symmetrical, the force polygon will be symmetrical about horizontal line 1-f,c.



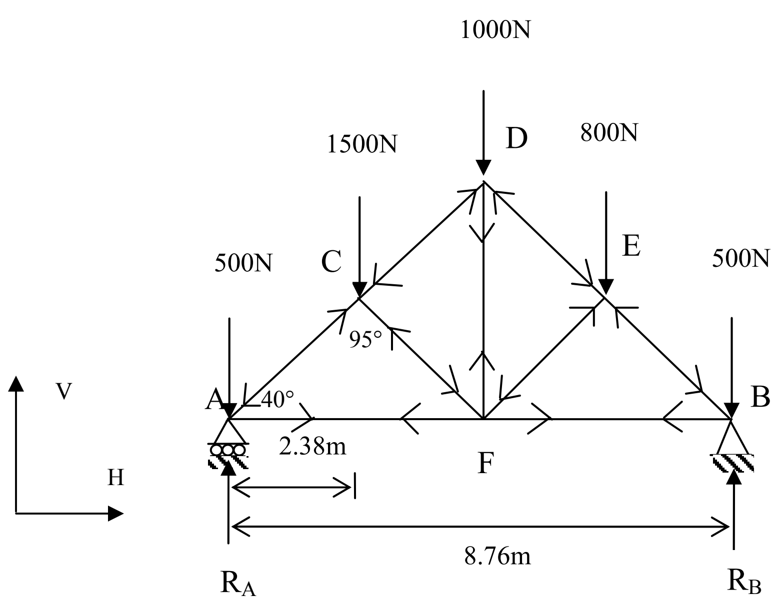

A few points are noteworthy:

1. The force polygon for a joint typically has an edge in common with the force polygon of another joint.
2. A member can have a zero force.
3. Under one set of loads a member can be under tension, but under another set of loads, the same member can be under compression.

2. The Algebraic Method: ( $\sum H = 0$ ; $\sum V = 0$ )

In the algebraic method of determining the forces in statically determinate trusses, also called the method of **joint resolution**, we first calculate the reactions by taking moments, then, as in the graphical method, we proceed joint-by-joint to a joint with only 2 unknown forces. At each joint we simply apply $\sum H = 0$ and $\sum V = 0$, making use of trigonometry.

This is best demonstrated by example. Consider the following symmetrical truss that is unsymmetrically loaded. Bow's Notation is not needed so we label the joints as shown.




As the loads are vertical we know that there is no horizontal reaction at B. To get the vertical reactions, take moments about A (TMA A). Consider an anti-clockwise moment as positive (+ve) and the positive directions of the forces are indicated by the coordinate system shown:

$\sum M=0$: $8.76R_B - (2.38x1500) - (4.38x1000) - (6.38x800) - (8.76x500) = 0$

$R_B = 1990.18$ N

$\sum V=0$: $R_A + R_B - 500 - 1500 - 1000 - 800 - 500 = 0$

$R_A = 500 + 1500 + 1000 + 800 + 500 - 1990.18 = 2309.82$ N

For all the calculations at a joint, initially assume that all the forces are pulling away from the joint. So if you get a -ve answer for a force, you know the force is in the next direction. As for the graphical method, when you solve for the force in a member at a joint, the force at the other end of the member is in the opposite direction.

Joint A:

$\sum V=0$: $F_{AC}\sin 40 - 500 + 2309.82 = 0$; $F_{AC} = -2814.65$ N
$\sum H=0$: $F_{AF} + F_{AC}\cos 40 = 0$; $F_{AF} = -(-2814.65) \times \cos 40 = 2156.02$ N

To choose the next joint, it must be one with only 2 unknown forces.

Joint C:

$\sum V=0$: $F_{CD}\sin 40 - F_{CF}\sin(180\text{-}95\text{-}40) - F_{CA}\sin 40 - 1500 = 0$
$F_{CD}\sin 40 - F_{CF}\sin(180\text{-}95\text{-}40) = F_{CA}\sin 40 + 1500$ .

Noting that $F_{CA} = F_{AC}$,
$0.643\ F_{CD} - 0.707\ F_{CF} = -2814.65\text{x}0.643+1500 = -309.82$ (1)

$\sum H=0$: $F_{CD}\cos 40 + F_{CF}\cos(180\text{-}95\text{-}40) - F_{CA}\cos 40 = 0$
$0.766\ F_{CD} + 0.707\ F_{CF} = F_{CA}\cos 40 = -2156.02$ (2)

$(0.643+0.766)\ F_{CD} = -309.82-2156.02= -2465.84$
$F_{CD} = -1750.07$ N

Sub in (1),

$0.643 \text{ x } -1750.07 - 0.707\ F_{CF} = -309.82$
$F_{CF} = -1153.43$ N

To choose the next joint, it must be one with only 2 unknown forces.

Joint D:

$\sum V=0$: $-F_{DE}\sin 40 - F_{DF} - F_{DC}\sin 40 - 1000 = 0$
$F_{DE}\sin 40 + F_{DF} = -\ F_{DC}\sin 40 - 1000 = 125.29$

$0.643\ F_{DE} + F_{DF} = 125.29$ (1)

$\sum H=0$: $F_{DE}\cos 40 = F_{DC}\cos 40$
$F_{DE} = -1750.07$ N

Sub in (1),

$F_{DF} = 125.29 + 0.643x1750.07$
$F_{DF} = 1250.59$ N

Joint E:

$\sum V=0$: $-F_{EB}\sin40 - F_{EF}\sin45 + F_{ED}\sin40 - 800 = 0$
$-0.643F_{EB} - 0.707F_{EF} = 1925.3$

$\sum H=0$: $F_{EB}\cos40 - F_{EF}\cos45 - F_{ED}\cos40 = 0$
$0.766\ F_{EB} - 0.707\ F_{EF} = -1340.55$

$(0.643+0.766)\ F_{EB} = -3265.85$
$F_{EB} = -2317.85$ N

$F_{EF} = (0.766 \text{ x } -2317.85 + 1340.55)/0.707 = -615.16$ N

Joint B:

$\sum H=0$: $-F_{BF} - F_{BE}\cos40 = 0$
$F_{BF} = 2317.85 \text{ x } 0.766 = 1775.47$ N

<u>Check: Given the possibility of human error in engineering hand calculations, results must be checked.</u> In this case we use the information at the support we did not start from and determine whether the sum of vertical forces is zero, as it must be.

$R_B - 500 + F_{BE}\sin40 = 1990.18 - 500 - 0.643x2317.85 = -0.197 \text{ N} \approx 0.$

The small difference is due to round-off error.

b. <u>Finding the Internal Forces (in Planar Trusses) by the Method of Sections:</u>

Another method for determining the forces in plane trusses is the "Method of Sections". This method is used when the forces in only a few members are required such as for checking the results of more laborious calculations, or getting the forces in members deemed to be under the maximum forces.

**IF #29 The Method of Sections is mainly used when only the forces in a few members are required.**

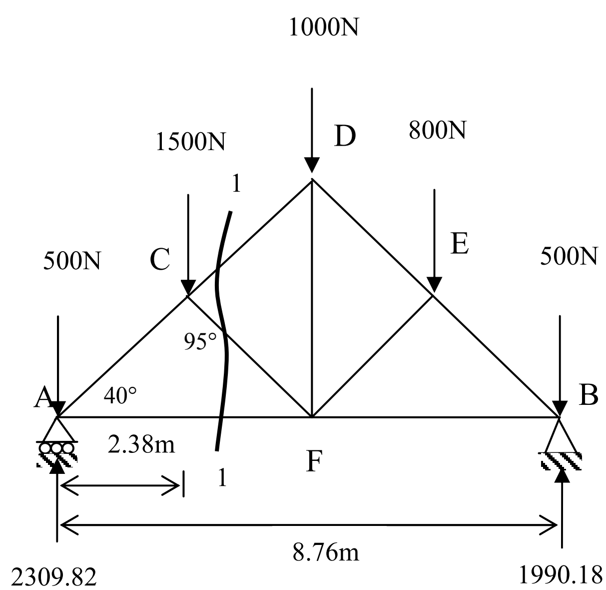
Like the other methods, the principle under which the Method of Sections operates is the laws and corollaries of equilibrium.

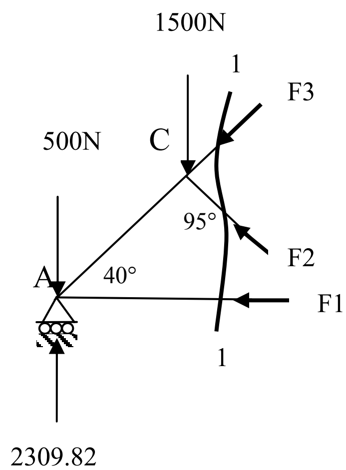
In this case, the truss is cut into 2 pieces and one piece examined. But since the truss must remain in equilibrium, at the points where the members are cut, forces are placed in the direction of each member. It is these forces that are calculated. These forces are inserted to represent the effect of the other piece in terms of what is required to maintain equilibrium of the whole structure. Therefore, these forces act as external forces on the piece. The cutting into 2 pieces is called "taking a section", hence the name of the method.

The steps involved in applying the Method of Sections are:

Step 1. Calculate the reactions as you would for the method of joint resolution.
Step 2. For the member concerned, take a section in such a way that, in addition to cutting the member, 2 other members are also cut but the lines of action of these 2 must intersect. The section need not be vertical.
Step 3. Select a piece (it is common to choose the piece with the fewer external loads) and label the forces at the cut members, as well as all the applied loads and reactions.
Step 4. Apply the equilibrium equation $\sum M=0$, by taking moments of all the forces (applied loads, reactions, and those inserted at the members), but about the point of intersection mentioned in step 2.
Step 5. Step 4 results in 1 equation with 1 unknown which is then easily solved.

As an example, refer to the previous problem. What is the force in member AF?






Taking moments about C: (The distances are determined from the geometry of the truss; take clockwise as +ve).

Clockwise: F1x2.38tan40 + 2309.82x2.38
Anti-clockwise: 500x2.38

Hence F1x2.38tan40 + 2309.82x2.38 = 500x2.38
1.997F1 = 500x2.38-2309.82x2.38 = -4307.37
F1 = -2156.92 N

This compares well with the joint resolution calculation of 2156.02 N. However the problem is not complete until we determine if the member is in tension or compression.

The negative sign for the calculated value of F1 indicates that actual sense of the force is opposite to that assumed for the calculation. And since F1 is an external force, this means that the member is being pulled, hence in tension. This is also in agreement with the calculation via joint resolution for the same member.

c. Finding the Internal Forces (in 3D Trusses) by the Method of Tension Coefficients:

In this section we complete our presentation of structural analysis methods for statically determinate plane trusses with the Method of Tension Coefficients.

In essence, the Method of Tension Coefficients is the same as the method of joint resolution presented previously, but applied to 3D pin-ended structures. Though rarely of practical application, it is thought necessary to familiarize students with the method in order to train thinking in 3D (after all, the 2D structures are merely idealizations), as well as to extend the intuition developed by studying the equilibrium of concurrent forces in 2D, to the 3D space.

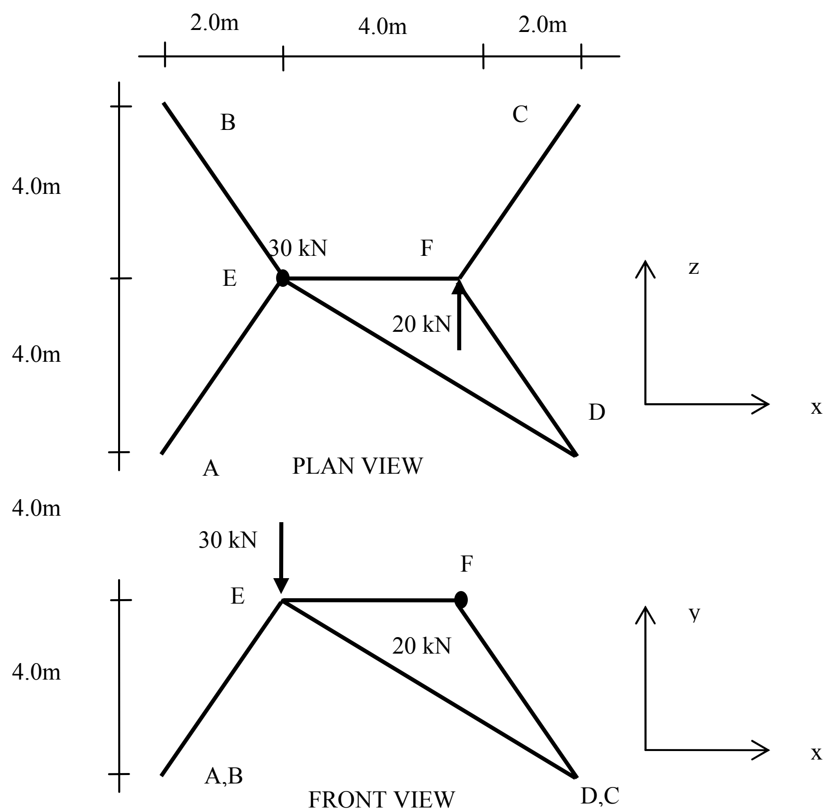
The **tension coefficient** is the force in the member divided by its 3D length. By using the tension coefficient it becomes much easier to build up the equilibrium equations without calculating sines and cosines.

For 3D trusses the equation expressing the condition of statical determinacy is m+r=3j. In this equation, there are 3 reactions per support (1 vertical and 2 horizontal), and 3 equilibrium equations per joint ($\sum V=\sum H_1=\sum H_2=0$). These result in simultaneous equations for the whole structure excluding the supports hence it is not necessary to first calculate reactions as you would for the joint equilibrium method.

To apply the Method of Tension Coefficients, the following steps are used:

Step 1. Draw the structure in plan and elevation views.
Step 2. Calculate the 3D length of the members ($L = \sqrt{(\Delta x^2+\Delta y^2+\Delta z^2)}$).
Step 3. For each joint assume that each member pulls away from the joint and represent that force by the tension coefficient (e.g. $T_{AB}$ for the tension coefficient of member AB).
Step 4. For each joint apply $\sum V=\sum H_1=\sum H_2=0$ using a convenient coordinate system, but instead of using trigonometry directly, refer to the views of the structure from step 1, and use the projected length of the member as the number in front of the tension coefficient.
Step 5. Solve the simultaneous equations arising from step 4, and get the force in the member by multiplying the calculated tension coefficient for the member by its 3D length.
Step 6. As for the method of joint resolution, determine if the member is in tension or compression.
Step 7. Determine the reactions at the supports from the member forces (using trigonometry).

Though apparently lengthy, the procedure is quite simple to perform. Consider the following example. The structure has 2 applied forces: 30 kN vertically downwards at joint E, and 20 kN horizontally (i.e. the z direction) at joint F.




Member lengths:

$AE = BE = CF = FD = \sqrt{(4^2+2^2+4^2)} = 6.0m$; $EF = 4.0m$
$DE = \sqrt{(4^2+6^2+4^2)} = 8.246m$

Joint E:

x: $-2T_{EA} - 2T_{EB} + 4T_{EF} + 6T_{ED} = 0$ (1)
y: $-4T_{EA} - 4T_{EB} - 4T_{ED} -30 = 0$ (2)
z: $-4T_{EA} + 4T_{EB} - 4T_{ED} = 0$ (3)

Joint F:

x: $-4T_{FE} + 2T_{FC} + 2T_{FD} = 0$ (4)
y: $-4T_{FC} - 4T_{FD} = 0$ (5)
z: $4T_{FC} - 4T_{FD} + 20 = 0$ (6)

Solving equations (1) to (6) we get:

$T_{EA} = T_{ED} = -15/8$
$T_{EB} = -15/4$
$T_{EF} = 0$
$T_{FC} = -2.5$
$T_{FD} = 2.5$

Hence,

$F_{EA} = -(15/8)x6 = -11.25$ kN
$F_{ED} = -(15/8)x8.246 = -15.461$ kN
$F_{EB} = -(15/4)x6 = -22.5$ kN
$F_{EF} = 0$
$F_{FC} = -2.5x6 = -15$ kN
$F_{FD} = 2.5x6 = 15$ kN

Vertical reactions:

Considering the angle of the member from the vertical and in the plane of the member,

$V_A = (4/6)x11.25 = 7.5$ kN
$V_B = (4/6)x22.5 = 15$ kN
$V_C = (4/6)x15 = 10$ Kn
$V_D = (4/8.246) \text{ x } 15.461 - (4/6) \text{ x } 15 = -2.5$ kN

# 4.0 STATICALLY DETERMINATE BEAMS – "REACTIONS THEN SECTIONS"

Beams are solid bodies that deform mainly by bending when loads are applied to them. If an engineer is to appropriately select a beam to resist the applied loads, then it is necessary to determine the stresses developed within the beam due to those loads.

Recall IF #14:

> "We may say that there are 3 kinds of balance of forces occurring in a structure – the balance of the total applied forces with the reactions at the supports; the balance of the forces occurring at each joint, and the balance of the internal forces at any section of a member with the forces acting on the member and at the ends of the member."

With respect to the underlined portion, recall also Fig. 3c reproduced below.

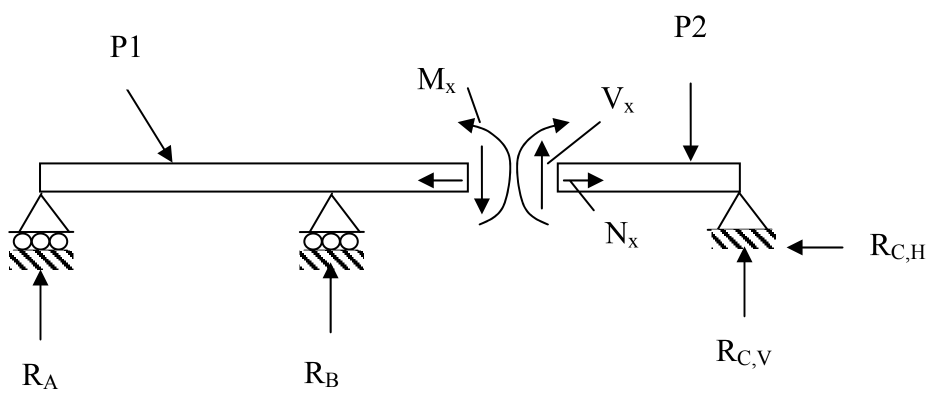



Fig. 4.1

The point to note here is that if you take a section anywhere along the beam you get 2 pieces. However to maintain the original equilibrium of the beam we must insert external forces at the ends of each of the 2 pieces, and these added forces must be in equilibrium with the applied loads and reactions. (Remember we did this also for the Method of Sections for trusses). The moment $M_x$ and vertical force $V_x$ are referred to as the **applied moment** and **applied shear force** at section x and since they are in equilibrium with the external loads, are a function of the applied loads (P) and reactions (R) only.

It is important to notice here that we can use either the left piece **OR** the right piece to calculate $M_x$ and $V_x$ by applying the statics equations $\sum M = \sum V = 0$ to that piece.

As was indicated in Chapter 2, the above beam is statically indeterminate therefore additional methods are required to calculate the unknown reactions and forces at the joints. However, once these are determined, we then have all the external forces on the beam so can calculate the $M_x$ and $V_x$ at any section by using the statics equations only.

The scope of our Mechanics of Solids course is limited to statically determinate beams therefore the beam is single-span and the unknowns are reactions only which are determined by statics only. In calculating the $M_x$ and $V_x$ for statically determinate beams, the sequence of calculations is therefore that we first calculate the reactions, then the section forces ($M_x$, $V_x$), hence the title of this chapter - "reactions then sections".

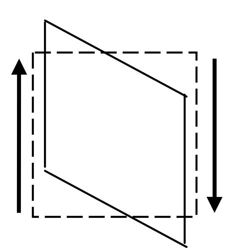
As the beam is a set of sections, the distributions of $M_x$ and $V_x$ along the length of the beam are our primary concern. The aim of this Chapter is to present graphical and algebraic methods for determining these distributions called the bending moment diagram (BMD), and shear force diagram (SFD), respectively

a. <u>Shear Force (V) and Bending Moment (M):</u>

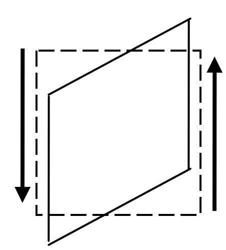
1.1 Sign Convention for V and M

Notice that at the section of the beam above there are 2 Ms and 2 Vs. Recall also that in Chapter 1 Section 1.c.1 (Fig. 1.6), we referred to a section as a 1D component. This is meant to represent the fact that at a section there are 2 equal and opposite forces representing the equilibrium of the structure to the left and right of the section.

At the section, the shear force V is tending to deform the material in the vicinity of the section in either of the following ways (the dashed line is the original undeformed shape):



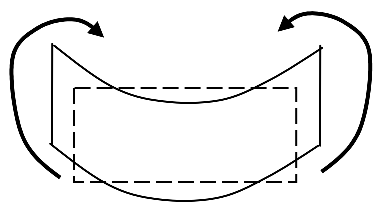
Positive $V_x$



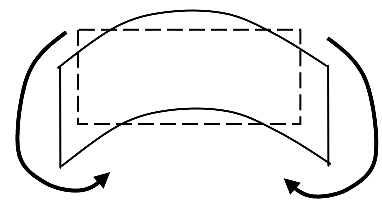
Negative $V_x$

The typical sign convention for a shear force is that if the shear deformation is as shown on the left, the shear force V causing this is called positive, but if as shown on the right, the shear force V is considered negative. This is usually encapsulated by the phrase "up to the left and down to the right is positive". Notice that shear is characterized by the change in angle at the corners of the box from being right angles in the original undeformed state.

At the section, the bending moment M is tending to deform the material in the vicinity of the section in either of the following ways (the dashed line is the original undeformed shape):



Positive $M_x$



Negative $M_x$

The typical sign convention for a bending moment is that if the bending deformation is as shown on the left (called a "sagging" deformation), the bending moment M causing this is called positive, but if as shown on the right (called a "hogging" deformation), the bending moment M is considered negative. This is usually encapsulated by the phrase "sagging is positive".

Looking at the shear again, and examining Fig. 4.1 above, a positive V is downwards if the rest of the beam is on the left, but also, a positive V is upwards if the rest of the beam is on the right.

Likewise, a positive M is anti-clockwise if the rest of the beam is on the left, but also, a positive M is clockwise if the rest of the beam is on the right.

**IF #30 The sign convention for V is: "up to the left and down to the right is positive". This also means that a positive V is downwards if the rest of the beam is on the left, but upwards if the rest of the beam is on the right.**

**IF #31 The sign convention for M is: "sagging is positive". This also means that a positive M is anti-clockwise if the rest of the beam is on the left, but clockwise if the rest of the beam is on the right.**

b. Shear Force Distribution at Section x, $V_x$, and Bending Moment Distribution at Section x, $M_x$ by the Graphical Method:

1. Equilibrium of an Infinitesimal Beam Length

If we take an infinitesimal length of a beam which is bending under an applied load q per unit length, this beam length must be in equilibrium. Hence q must be in equilibrium with V and M at both ends of the length. Now because the beam length is deforming, the V and M at one end, must be different from the V and M at the other end.

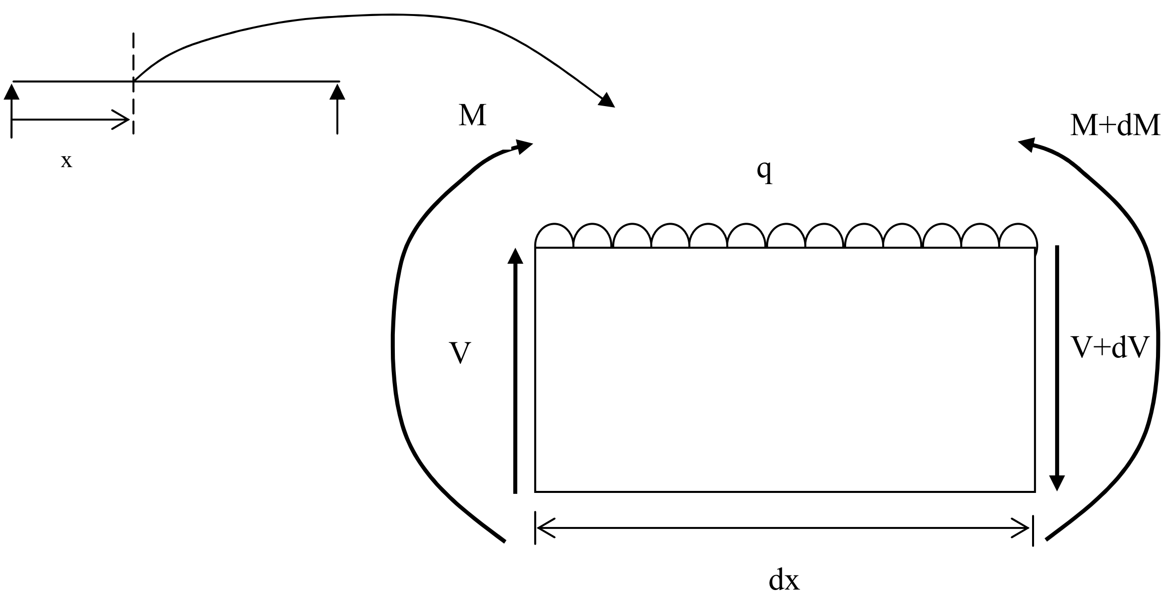



Fig. 4.2

It is usual to take the origin of the coordinate system that defines the location of a section (i.e. x) as the left end of the beam. This is because it is consistent with the definition of positive V and M .

For equilibrium $\sum M=0$ and $\sum V=0$.

$\sum M=0$:
Taking moments about the right edge,
$M - (M+dM) + V\,dx - q\,dx\,(dx/2) = 0$
Neglecting higher order quantities, the fourth term is eliminated and we get,
$-dM + V\,dx = 0$, hence
$V = dM/dx$
$dM = V\,dx$

$$M = \int V\,dx \qquad (1)$$

$\sum V=0$:
$V - q\,dx - (V+dV) = 0$, hence
$-q\,dx - dV = 0$
$dV = - q\,dx$

$$V = \int -q\,dx \qquad (2)$$

2. The Shear Force Distribution ($V_x = - \sum$ forces q to left of x)

In equations (1) and (2), the integral is a definite integral with the upper limit as x, and the lower limit as 0. Since an integral is a summation, equation (2) can be re-written as,

$$V_x = - \sum \text{forces } q \text{ to left of } x \qquad (3)$$

Equation (3) can therefore be plotted to scale and the result defines the shear force distribution.

3. The Bending Moment Distribution ($M_x$ = Area of SFD to the left of x)

As the integral of a variable y with x equals the area under the curve y=f(x), then from equation (1), the area below the shear force distribution, enables the calculation of the bending moment distribution. Hence equation (1) can be re-written as,

$$M_x = \text{Area of SFD to the left of } x \qquad (4)$$

Hence given a plot of the SFD, the BMD can be easily derived simply by calculating areas.

A noteworthy point is that from equation (1), $dM/dx = V$. But we know from calculus that when $dy/dx=0$, the value of y is at its maximum value. Hence at the point on the beam where V=0, this is also the point of the maximum bending moment in the BMD.

Lastly, the point on the BMD where M=0 is called the **point of contraflexure** and has a number of uses: (1) for devising approximate solution methods for statically indeterminate beams, (2) for determining the best location for joints in beams, and (3) for determining when to switch the location of steel in a reinforced concrete beam from the top to the bottom face of the beam.

**IF #32 The shear force at a section x can be determined graphically from, $V_x = -\sum$ forces q to left of x, where x is measured from the left end of the beam.**

**IF #33 The bending moment at a section x can be determined from the SFD, by using $M_x$ = Area of SFD to the left of x, where x is measured from the left end of the beam.**

**IF #34 At the point on the SFD where V=0, the bending moment is at its maximum value in the BMD.**

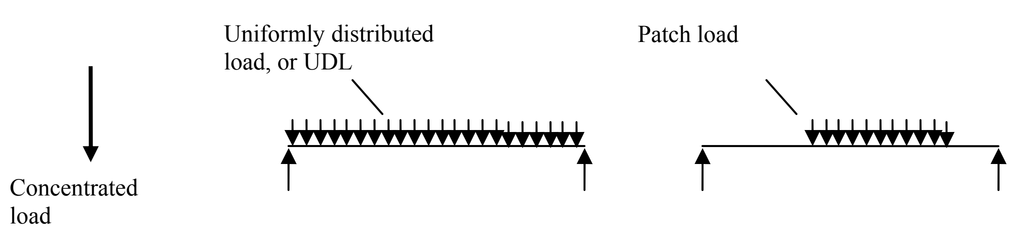
**IF #34 The point of contraflexure is the point on the beam where for a given set of applied loads, the bending moment is zero.**

c. <u>Shear Force Distribution at Section x, $V_x$, and Bending Moment Distribution at Section x, $M_x$ by the Algebraic Method:</u>

In the last section the determination of the shearing force and bending moment diagrams by graphical methods was presented. In this section the same objective is achieved by algebraic methods.

Before presenting the procedure, the following are important considerations.

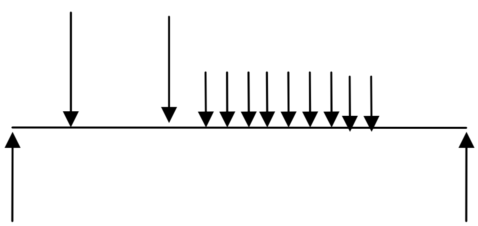
Common Load Types:



The concentrated load can be applied anywhere on the beam, as can the patch load. Another symbol for a UDL is that used in Fig. 4.2. The UDL and patch loads are stated per m (e.g 5 kN/m). Any number, location, or combination of these loads can be applied to the beam.

Classification of Load Arrangement:

If more than one concentrated load is applied, or a patch load is applied, the arrangement is called **discontinuous loading**. Hence **continuous loading** refers to a loading arrangement involving only 1 concentrated load and/or a UDL. An example of discontinuous loading is as follows.



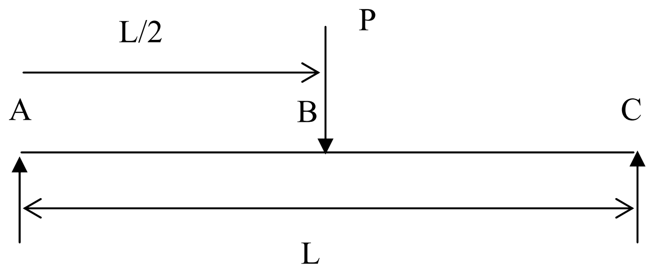
Example of discontinuous loading

The procedure of determining the SFD and BMD by the algebraic method is as follows:

Step 1. Setup a x-y coordinate system at the left end of the beam with positive x going to the right, and positive y going upwards; x defines the position of a section.
Step 2. Determine the reactions at the supports by taking moments.
Step 3. Cut a section x-x anywhere between the left support and the first applied load on the left.
Step 4. Separate out the left piece of the beam and insert the external moment and shear, in their positive directions (anti-clockwise for M; downwards for V), at the right end.
Step 5. For that portion of the beam, apply the statics equation $\sum V = 0$ and make V the subject of the equation. Similarly, apply $\sum M = 0$ and make M the subject of the equation. These equations are the equations for V and M in that portion of the beam only.
Step 6. If there is another applied load to the right of the applied load of step 4, cut a new section in the portion of the beam between them and repeat steps 4 and 5.
Step 7. For each successive portion of the beam between loads do as in step 6.

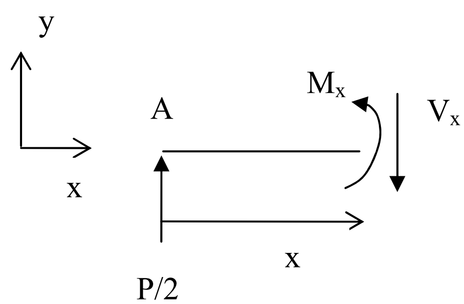
Consider the following application of the procedure to derive 2 standard cases. A standard case is one that student is expected to know the formula for by rote.

Example 1. A simply supported beam with a concentrated load P at mid-span.




$R_A = R_C = P/2$

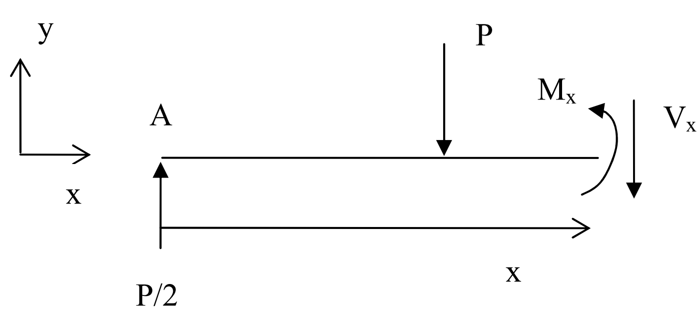
<u>For x between 0 and L/2:</u>




$\sum V = 0$:
$P/2 - V_x = 0$
$V_x = P/2$

$\sum M = 0$:
$M_x - Px/2 = 0$
$M_x = Px/2$

<u>For x between L/2 and L:</u>

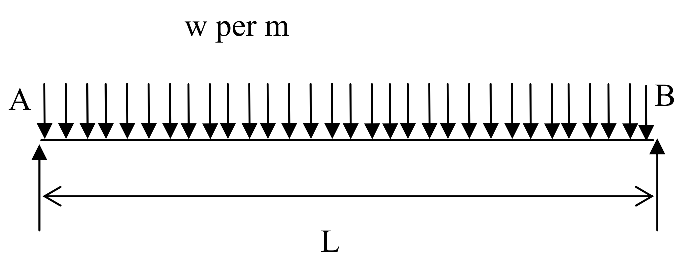


$\sum V = 0$:
$P/2 - P - V_x = 0$
$V_x = -P/2$

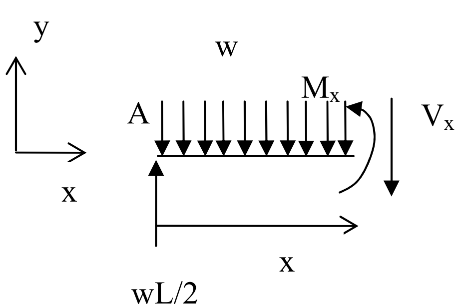
$\sum M = 0$:
$M_x - Px/2 + P(x-L/2) = 0$
$M_x = Px/2 - Px + PL/2 = P(L-x)/2$

At $x = L/2$, $M_x = P(L-(L/2))/2 = PL/4$

Example 2. A simply supported beam with a UDL of w per m.




$R_A = R_B = wL/2$




$\sum V = 0$:
$wL/2 - wx - V_x = 0$
$V_x = wL/2 - wx = w(L/2 - x)$

$\sum M = 0$:
$M_x - wxL/2 + wx^2/2 = 0$
$M_x = wx(L-x)/2$

At V = 0, $M_x$ is at a maximum. Hence from the equation for $V_x$, x = L/2. Hence the maximum moment in a simply-supported beam under a UDL is,

$M_{x,max} = w(L/2)(L-L/2)/2 = w(L/2)(L/2)/2 = wL^2/8$