9.0 DEFLECTION, SLOPE OF BEAM CROSS-SECTIONS

The aim of this section is the calculation of the deflection and slope at any section of a beam. The ability to perform these calculations is important for the following reasons:

1. In our Mechanics of Solids course we have limited our attention to beams that are statically determinate. However, practical construction demands the use of redundant structures. As we saw in Chapter 2, to solve for the unknown forces in redundant structures, additional equations are required and these are provided by considering the way the that the ends of the beams deform at the joint. This is in terms of the deflection and slope. This then enables the development of the two principal approaches to solving for redundant structures and in one of these, called the **flexibility method**, the focus is on using information about the deflections and slopes to set up the main equations to be solved. The other method is called the **stiffness method**. These methods are the basis of most of what we learn later in the UWI Civil and Environmental Engineering programme when we study Structural Mechanics and Structural Analysis.

2. The performance of a structure is considered satisfactory not only when the structure safely resists the loads applied to it. Another important consideration is the confidence of the user of that structure that all is well with the structure. If it deforms too much, that confidence is lost, even if the engineer knows that the structure is safe. Hence a key aspect in the design of a structure is that its deformations, hence deflections, are within acceptable limits. This is called an aspect of the **serviceability** of the structure.

3. The proper performance of structures to earthquakes requires that attention be focused on the displacements of the members of structure during the earthquake, rather than the forces in those members.

a. The Moment-Curvature Relationship

From the Engineer's Beam Theory of Chapter 6 we know:

$M_x/I = E/R$

Hence

$M_x = EI \times (1/R)$ (9.1)

Consider an infinitesimal length of the deflected beam:

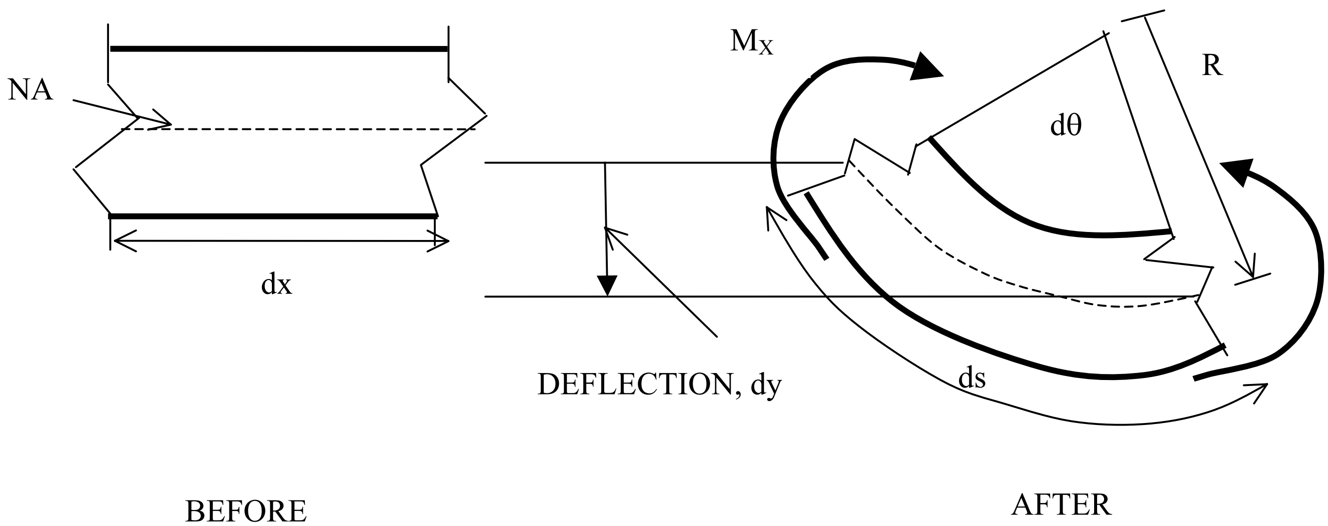


Fig. 9.1 Deflection of an Infinitesimal Length of a Beam

The length ds is the infinitesimal length of the fiber at the neutral axis.

$ds = R\, d\theta$

Hence

$1/R = d\theta/ds$.

But for small deflections, $\theta \approx dy/dx$ and $ds \approx dx$, therefore

$1/R = d^2y/dx^2$.   Hence, equation (9.1) becomes

$$M_x = EI\,(d^2y/dx^2) \qquad (9.2)$$

y is the deflection of the beam at point x; dy/dy (=θ) is the slope of the beam at point x, and d^2y/dx^2 is the curvature of the beam at point x.

b. Double Integration Method

Given equation (9.2), the slope and deflection at point x are therefore obtained by integration of the M_x /EI expression for the moment at point x.

$d^2y/dx^2 = M_x / EI$

Slope, $\theta = dy/dx = \int (M_x / EI)\, dx + P$ (9.3)

Deflection, $y = \iint (M_x / EI)\, dx + Px + Q$ (9.4)

To use equations 9.3 and 9.4 for solution, an expression for M_x is derived for the given problem, and substituted in equation 9.3. This is then integrated once and the boundary conditions of the problem are used to determine the integration constant P. This completes the calculation of the slope at x.

The deflection at x is then obtained by integrating the slope equation and using the boundary conditions of the problem to determine the integration constant Q.

The double-integration method can be used to determine the deflection at a point for any problem. However, recall from Chapter 4 Example 1, that to solve that problem we needed an expression for M_x from A to P, and another expression for M_x from P to B. Hence in using the Double-Integration Method for deflections calculations can be very tedious as the integrations must be performed for each M_x expression of each region between the applied loads, if the loading is discontinuous.

In such cases it is much easier to modify the way the loading is represented before performing the integration. This is the basis of the Macaulay Method of section d.

In the practical solution of deflection problems it is also much more convenient to use the deflection solutions for standard simple load conditions, and employ the Principle of Superposition.

The Double-Integration Method is therefore only efficiently used for such simple cases because the solutions for sets of P and Q are then not required.

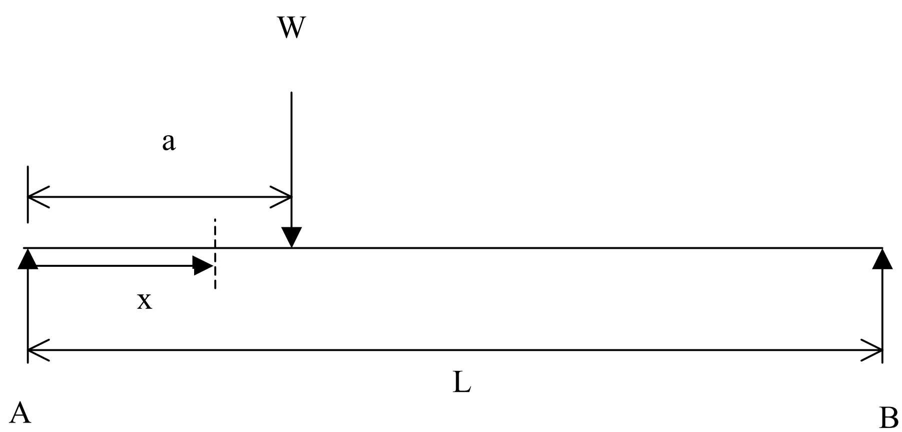
IF #47 Though the double integration can be used to solve for deflections and slopes for all beams, it is not the most efficient method except for simple cases.

c. <u>Standard Cases</u>

In the previous section it was stated that the solution of the deflection problem for simple load conditions can be used along with the Principle of Superposition to solve more complex problems.

We now employ the Double-Integration Method to solve some of these simpler problems. These solutions are called standard cases, and it is expected that the student will commit to memory these standard cases, and the others presented in the table.

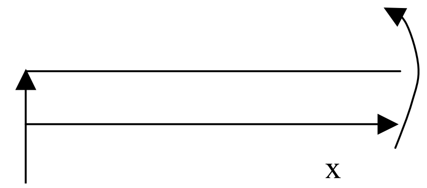
Case 1: Point Load Distance "a" From Left End:



Sign Convention: Upward deflection is +ve; sagging moment is +ve

TMA A, $Wa = R_B \times L$ $\quad R_B = Wa/L$
$R_A + R_B = W$
$R_A = W - R_B = W - Wa/L = W(1-a/L) = W(L-a)/L$

Take a section between AB distance x from A such that $x < a$:



Step 1: From above FBD, at x determine the moment equation:

$M_X = Wx(L-a)/L$

Step 2: Substitute in the deflection equation and integrate twice:
$EI\ (d^2y/dx^2) = M_X = Wx(L-a)/L$
$EI\ (dy/dx) = Wx^2(L-a)/2L + P$ (1)
$EI\ y = Wx^3(L-a)/6L + Px + Q$ (2)

Step 3: Apply boundary conditions:

At $x = 0$, $y = 0 \Rightarrow Q = 0$

The integration constant P depends on the location "a" of the point load.

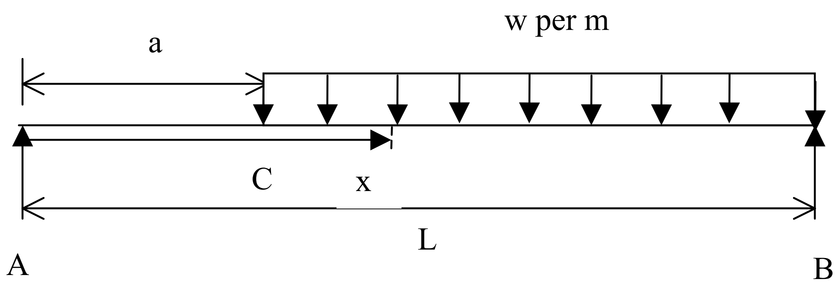
Max deflection for point load at center:

For a point load at the center $a = L/2$, and for the maximum deflection $x = L/2$
At $x = a = L/2$ $dy/dx = 0$. Substitute in (1)
$\Rightarrow EI\ (dy/dx) = Wa^2(L-a)/2L + P = 0 \Rightarrow P = -Wa^2/4$
Substitute in (2),
$\Rightarrow EI\ y = Wa^3(L-a)/6L - Wa^3/4 = WL^3/96 - WL^3/32 = \underline{-WL^3/48}$

Slope at support for point load at center:

$x = 0$ and $a = L/2$. Substitute in (1),
$\Rightarrow EI\ (dy/dx) = Wx^2(L-a)/2L - Wa^2/4 = \underline{-WL^2/16}$

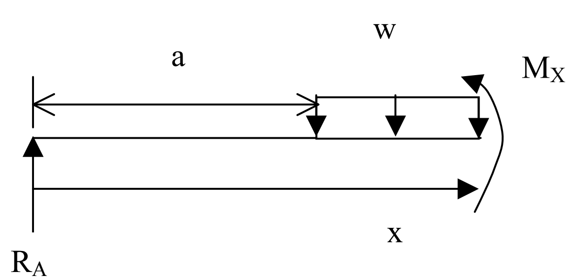
Case 2: UDL Load Starting Distance "a" From Left End:



Sign Convention: Upward deflection is +ve; sagging moment is +ve

TMA B, $R_A\ L = w(L-a)(L-a)/2$
$R_A = w(L-a)^2/2L$

Take a section between CB distance x from A:



<u>Step 1</u>: From above FBD, at x determine the moment equation:

$M_X + w(x-a)^2/2 = wx(L-a)^2/2L$
$M_X = wx(L-a)^2/2L - w(x-a)^2/2$

<u>Step 2</u>: Substitute in the deflection equation and integrate twice:

$EI\ (d^2y/dx^2) = M_X = wx(L-a)^2/2L - w(x-a)^2/2$
$EI\ (dy/dx) = wx^2(L-a)^2/4L - (wx^3/6 - wx^2a/2 + wa^2x/2) + P$ (1)
$EI\ y = wx^3(L-a)^2/12L - wx^4/24 + wx^3a/6 - wa^2x^2/4 + Px + Q$ (2)

<u>Step 3</u>: Apply boundary conditions:

At $x = 0,\ y = 0 \Rightarrow Q = 0$
At $x = L,\ y = 0$
$\Rightarrow P = -wL^2(L-a)^2/12L + wL^3/24 - wL^2a/6 + wa^2L/4$

$= - wL^3/24 – wa^2L/12 + wa^2L/4$ (3)

<u>Maximum deflection for UDL over whole beam</u>:

For udl over the whole beam,
$a = 0$
Hence by substitution in (3),
$\Rightarrow P = - wL^3/24$

By inspection the maximum deflection occurs at $x = L/2$. Substitute in (2) with $a = 0$,
$\Rightarrow EI\ y = w(L/2)^3L^2/12L – w(L/2)^4/24 - wL^3(L/2)/24$
$= wL^4(1/96 – 1/384 – 1/48)$
$= -5wL^4/384$

<u>Slope at support for UDL over whole beam</u>:

At the support $x = 0$. Substitute in (1) with $a = 0$,
$\Rightarrow EI\ (dy/dx) = P = -wL^3/24$

Note that in the following table, W is "big w" which is of force units, and not "little w" which is of force per length units. (i.e.W=wL); each beam is of length L, and the signs of the slope and deflection are ignored.

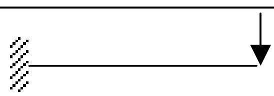
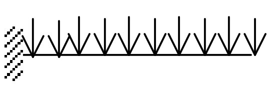
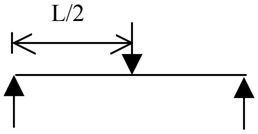
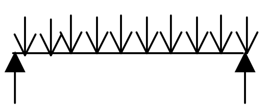
TYPE OF BEAM AND LOADING	MAX. SLOPE	MAX. DEFLECTION	SLOPE FACTOR	DEFLECTION FACTOR
	1/2	1/3	$WL^2/(EI)$	$WL^3/(EI)$
	1/6	1/8	“	“
	1/16	1/48	“	“
	1/24	5/384	“	“

Table 9.1 Slope and Deflection Formulae for Standard Cases

As an example on the use of Table 9.1, the slope at the end of a simply supported beam with a point load at the center is given by $WL^2/(16EI)$, and the maximum deflection in a cantilever with a point load at the end is given by $WL^3/(3EI)$.

d. Macaulay Method

In section b above it was mentioned that for many practical problems, in which case the loading is discontinuous, the Double-Integration Method usually is too tedious since a separate expression is required for M_x for each region between the loads. This means that we have to solve for the integration constants P and Q several times. This inconvenience was resolved by W. H Macaulay in 1919.

IF #48 The Macaulay Method is the most efficient way to calculate the slopes and deflections in a beam. It is also called the singularity or step function method.

The Macaulay Method enables the double integration, hence determination of P and Q, to be performed only once, even if the loading is discontinuous. The Macaulay Method achieves this by redefining the way the load is represented. This is done by mathematically transforming a load from its original discontinuous form as a function of x, to a continuous form as a function of x via the use of the **singularity function**.

The following are singularity function representations relevant to point loads and udl's.

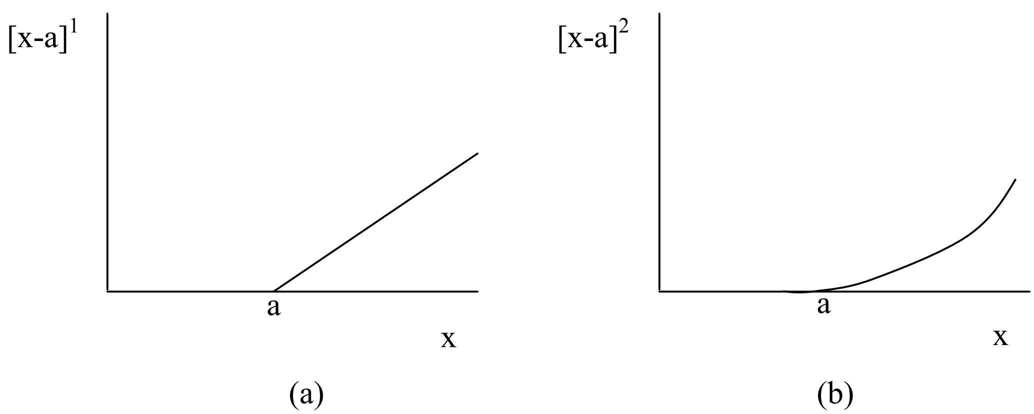


Fig. 9.2 Commonly Used Singularity Functions

The singularity function has the form,

$$f_n(x) = [x-a]^n \qquad (9.5)$$

It is important to note that the term on the right-hand side of equation (9.5) is a notation used to represent the functions shown graphically in Fig. 9.2. This is emphasized by the use of the square brackets [ ] in which case the variables are not to be treated algebraically. When the function is evaluated, then round brackets ( ) are used and you can then treat the variables algebraically.

In equation 9.5, when $n \geq 0$, the function evaluates to zero for $x < a$ but evaluates to $(x - a)^n$ for $x > a$. Its integral is,

$$\int_0^x [x-a]^n \, dx = [x-a]^{n+1} / (n+1) \qquad (9.6)$$

IF #49 The formula for integrating a singularity function when $n \geq 0$ is

$$\int_0^x [x-a]^n \, dx = [x-a]^{n+1} / (n+1)$$

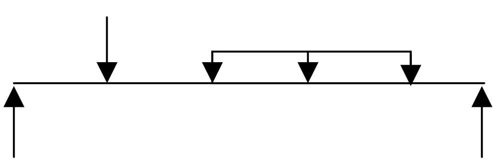
When forming the expression for M_x in a problem with a point load W, for any section in the region to the right of the point load the point load will contribute a moment of W[x-a] if a is the distance from the left end, to the point load. In this case, one can use the singularity function represented in Fig. 9.2 (a). Likewise, the one can use the singularity function represented in Fig. 9.2 (b) for a udl which starts "a" from the left end. This is because, the udl will contribute a moment of $w[x-a]^2/2$.

Therefore by the use of singularity functions loads that are discontinuous can be represented by continuous functions. This enables a procedure to be derived to perform the double-integration once and obtain the deflection. The procedure is as follows:

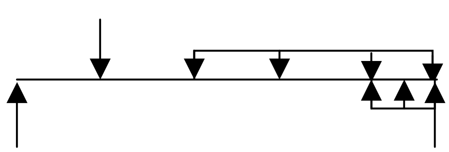
Rules for Using Macaulay's method:-

1. Select an origin at one end – usually left end.

2. Set the problem in proper form if needed – any udl in the space between last load from left, and the rightmost support must be made continuous to the right end, which means introducing a negative udl (i.e. load upwards) to cancel it.



becomes



3. Take a section in the region of the beam closest to the right support and determine the moment equation just as in the double integration method. However, for each applied load express its contribution to M_x in terms of the singularity function.

i.e. For point load distance "a" from left end: $w \Rightarrow w[x-a]$
For a udl starting distance "a" from left end: $w \Rightarrow w/2[x-a]^2$

4. Continue just as in the usual double integration method but

   a. When integrating a singularity function use equation 9.6:

      Eg. $\int [x-a]\, dx = \frac{1}{2} [x-a]^2$ and $\int [x-a]^2\, dx = 1/3\, [x-a]^3$

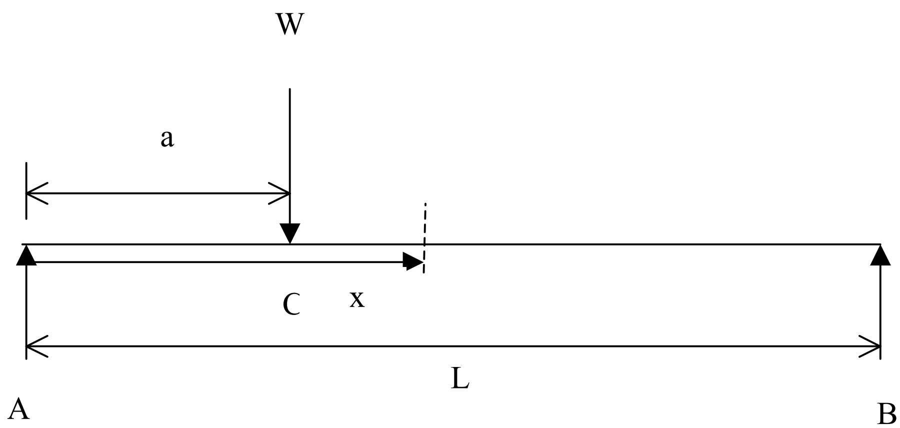
   b. When evaluating the step function replace the [ ] with ( ).

   c. Ignore values for (x-a) in any region where $x < a$

IF #50 When the moment equation is obtained in terms of a singularity function and the equation is being evaluated, ignore values for (x-a) where x < a.

Example 9.1:-

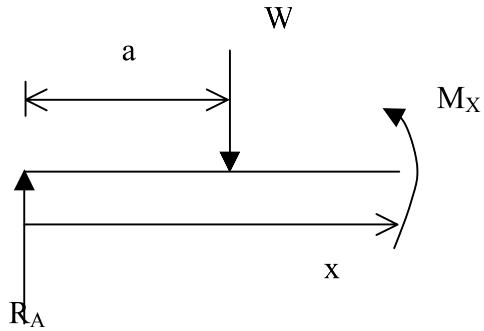
Use Macaulay's Method to redo Case 1 in section c above, to find the deflection under the point load of a simply-supported beam, if the point load is located at mid-span.



By taking moments, we know that

$R_A = W(L-a)/L$
$R_B = Wa/L$

Using Macaulay's Method, we must take a section in the region of the beam closest to the right support (Rule 3).



Moment equilibrium equation:

Singularity function due to W

$M_x - R_A x + W[x-a] = 0$
$M_x = R_A x - W[x-a]$

$EI\, d^2y/dx^2 = M_x = R_A x - W[x-a]$

Integrating to get slope equation,

Rule 4a for integrating the singularity function

$EI\, dy/dx = R_A x^2/2 - W[x-a]^2/2 + P$

Integrating again to get deflection equation,

$EI\, y = R_A x^3/6 - W[x-a]^3/6 + Px + Q$ (1)

Applying boundary conditions to equation (1):
When x = 0, y = 0, but in accordance with rule c, as x < a , the term with the singularity function is ignored. Hence Q = 0.

At x = L, y = 0, therefore from equation (1) and knowing that Q = 0,

Rule 4b: ( ) brackets as we are evaluating the singularity function

$0 = R_A L^3/6 – W(L-a)^3/6 + PL$

Hence making P the subject of the formula, noting that $R_A = W(L-a)/L$,

$P = - Wa(L-a)(2L-a)/(6L)$ (2)

Substituting (2) in (1) and noting that Q = 0, we get,

$y = -Wx\,(L-a)(2aL - a^2 - x^2)/(6EIL) - W[x-a]^3/(6EI)$ (3)

Equation (3) is the deflection y at any point x along the beam. Since it is based on Macaulay's Method then by Rule 4c, the second term on the right-hand side of (3) is to be ignored if $x < a$.

From (3), when $x = a$,

$y = -Wa^2\,(L-a)^2/\,(3EIL)$

If the load is at mid-span, $a = L/2$ hence,

$y = -WL^3/48EI$

10. TORQUE OF SHAFTS WITH CLOSED-END SECTIONS

Thus far in the presentation of the subject matter of Mechanics of Solids, we considered the axial and bending forces in one-dimensional members, followed by axial and bending stresses in those members. In those cases the stresses are longitudinal relative to the axes of the member, whether they are axial or bending stresses.

However, in practical construction shearing stresses can also be a controlling factor and it becomes important to have some insight into the nature of such stresses. The study of torsion provides an avenue for the study of shear stress.

In building construction there is the case of the external beams supporting a floor. In this instance the beam experience significant torsion as the floor it is supporting is on one side and the ends of the beam are rigidly attached to the columns at the ends. Steel beams of all types are prone to torsional movement when under vertical load with the effect of reducing the moment capacity if the length of the beam exceeds a certain amount.

In the study of torsion the member under the torsional or twisting moment is called a **shaft** but it should be noted that any predominantly one-dimensional member can experience torsion.

a. The Constitutive Relation for Shafts

Consider a rod of circular section and composed of homogeneous and isotropic material. In the following, the torque is the force, and the twist is the associated deformation of the shaft.

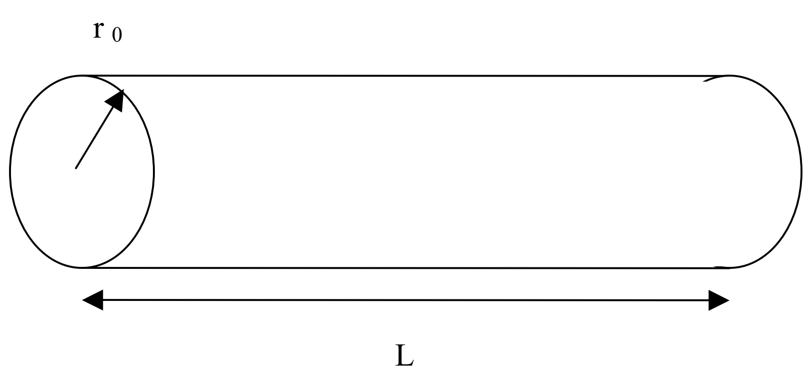


Fig. 10.1 Shaft Dimensions

If one end is fixed and a twist θ is applied at the other end, it can be shown that:

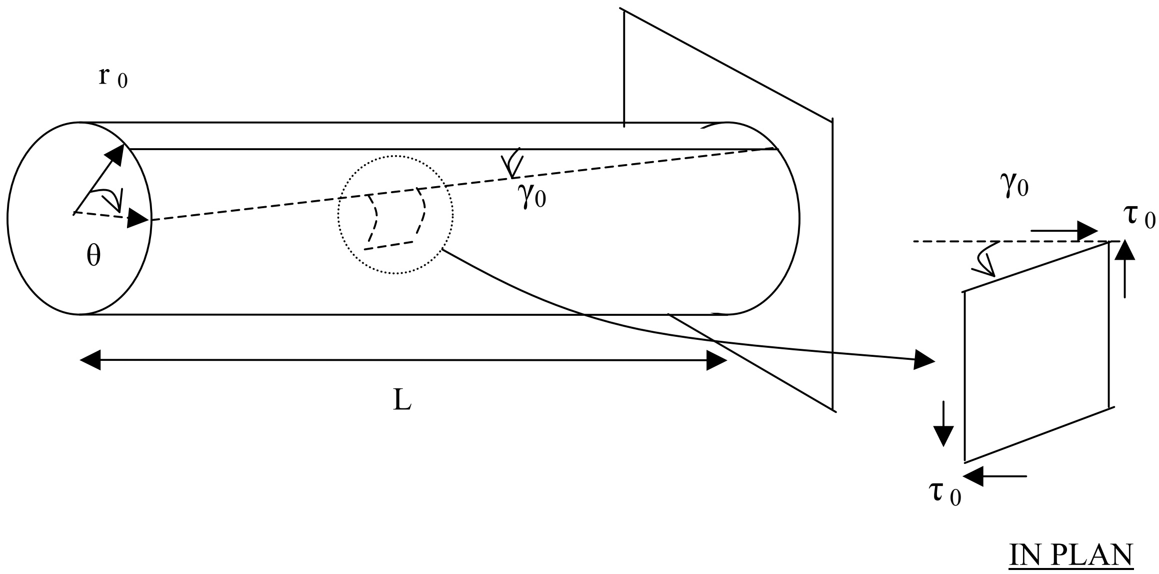


Fig. 10.2 Deformation of a Shaft

- Plane sections remain plane, and
- A radial line remains straight.

Any point on the edge of the rod will rotate through an angle γ relative to its original location.

Hence, $r_0 \theta = L \gamma_0$ (10.1)

Any element on the surface of the rod will be in a state of pure shear, with shear strain γ_0

Hence, $\gamma_0 = \tau_0 / G$ (10.2)

where τ_0 is the shear stress and G is the shear modulus.

From (1),

$$\gamma_0 = r_0 \theta / L = \tau_0 / G$$

Therefore,

$$\tau_0 / r = G \theta / L \quad (10.3)$$

The rod can be considered as a large number of thin cylinders bonded together.

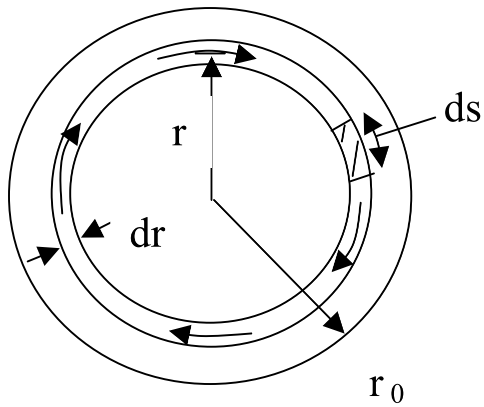


Fig. 10.3 Forces on an Infinitesimal Cylinder in a Shaft

For any of the thin cylinders of thickness dr at radius r, the force on an element of length ds = τ ds dr

Therefore the moment due to the force about the centre of the rod = τ r ds dr

Hence for the whole circumference of the thin cylinder, the torque on the cylinder,

$dT = \tau\, r\, (2\pi r)\, dr = 2\pi\, \tau\, r^2\, dr$

Therefore for all the thin cylinders from r = 0 to r_0, the total torque T,

$$T = \int_0^{r_0} 2\pi\, \tau\, r^2\, dr$$

From (3), $\tau = (G\,\theta)\, r / L$

Hence,

$$T = \frac{G\,\theta}{L} \int_0^{r_0} 2\pi\, r^3\, dr \qquad (10.4)$$

The integral in (10.4) is the polar moment of inertia, I_p

Hence, $T = G\,\theta\, I_p / L$

Hence from (10.3) we get,

$$T / I_p = G\theta / L = \tau_0 / r_0 \qquad (10.5)$$

Equation (10.5) is the constitutive relation for shafts.

IF #51 The constitutive relation for shafts under torsion is $T / I_p = G\theta / L = \tau_0 / r_0$

For a circular rod of solid cross-section,

$I_p = \pi r_0^4/2$

For a circular rod of hollow cross-section,

$I_p = \pi (r_0^4 - r_i^4) / 2$

where r_0 is the outer radius and r_i is the inner radius.

b. Concentric Shafts

A concentric shaft is one whose section is composed of 2 or more shafts bonded together. Concentric shafts are typically of different materials making the section a composite section.

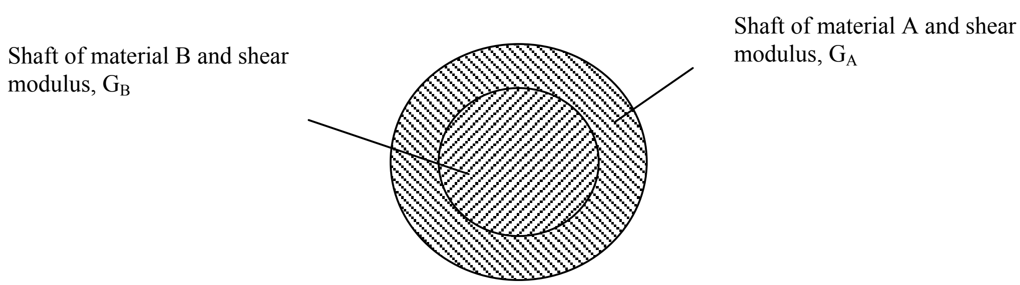


Fig. 10.4 Example of a Concentric Shaft

For concentric shafts under a torque T, the twist θ experienced by each material will be the same

From equation 10.5, $\theta = TL / (GI_p)$

Hence, as the shafts are of the same length,

$$\theta = T_A / (G_A I_{p\,A}) = T_B / (G_B I_{p\,B}) \qquad (10.6)$$

If the 2 shafts are of the same material, $G_A = G_B = G$ hence equation 10.6 becomes

$T_A / I_{p\,A} = T_B / I_{p\,B}$ (10.7)

Also, for concentric shafts, the torque T is shared between the 2 shafts so,

$T = T_A + T_B$ (10.8)

IF #52 For concentric shafts in torsion the twist angle is the same for all materials, and the total torque is the sum of the torque on each shaft.

c. Shafts in Series

If 2 shafts of different radius or material are placed in series and a torque applied to the assembly, the torque experienced by each shaft, hence the assembly, is the same but the twists in each material will be different.

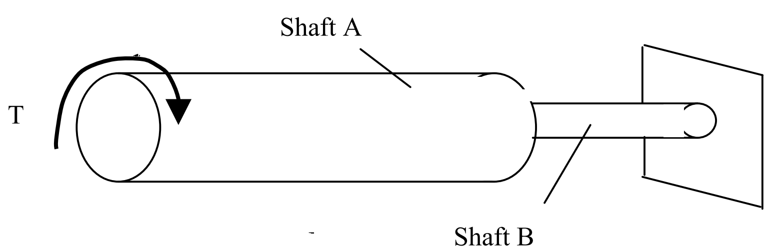


Fig. 10.5 Example of Shafts in Series

Hence,

$T = T_A = T_B$ (10.9)

Therefore, it can be shown that for shafts in series each of the same material and r_0, the strength of the assembly is determined by the strength of the shaft with the lower I_p.

IF #53 For shafts in series the twist angle is different for each shaft, and the total torque is the same on each shaft.

d. Efficiency of Shafts

The issue here is that we have 2 shafts – one hollow and one solid. If for the same cross-sectional area of the same material, and experiencing the same maximum shear stress, will the hollow shaft be able to carry a higher torque than the solid shaft, or vice versa?

Since the 2 shafts experience the same shear stress, then from equation 10.5

$T_H\, r_0 / I_{pH} = T_S\, r / I_{pS}$ (10.10)

$T_H / T_S = (r / r_0) (I_{pH} / I_{pS})$ (10.11)

But, $I_p = \pi r^4/2$ for the solid shaft,

$I_p = \pi (r_0^4 - r_i^4) / 2$, for the hollow shaft

Hence, $I_{pH} / I_{pS} = (r_0^4 - r_i^4)/ r^4$

So 10.11 becomes,

$T_H / T_S = (r_0^2 + r_i^2) / (r_0\ r)$ (10.12)

Let $r_0 / r_i = k$

Hence 10.12 becomes

$T_H / T_S = (k^2 + 1) / [k \sqrt{(k^2 - 1)}]$ (10.13)

As k is always > 1, the hollow shaft will always be able to carry a higher torque than the solid shaft (for all properties other than radii being the same).

IF #54 The ratio of the torsion capacities of a hollow vs a solid shaft is given by $T_H / T_S = (k^2 + 1) / [k \sqrt{(k^2 - 1)}]$ where $k = r_0 / r_i$.

e. The Power-Torque Relationship

The power-torque relationship is given by,

$T = 60\, P/ (2 \pi N)$ (10.14)

where,

T is in units of Nm

P is the power in Watts (i.e. Nm / second)
N is the number of revolutions per minute

Also,

$$P = T\,\omega \qquad (10.15)$$

where,

ω is the angular velocity in units of radian / second

And, the work done per revolution in Joules = T θ

$$\text{Work done per second} = 60\,T / (2\,\pi\,N) = T^2 / P \qquad (10.16)$$

IF #55 The power-torque relationship is given by $T = 60\,P / (2\,\pi\,N)$; T is in Nm, P in watts, and N in revs per minute.

11.0 STRESS, STRAIN IN 3D "BULK MASS" SOLIDS

In Chapter 1, we defined the term "solids" as the set of 3D objects termed "frameworks" and "bulk masses". From Chapters 2 to 10 the focus has been on 1D components of frameworks - ties/struts and beams, mostly as part of statically determinate systems. In this section we conclude the presentation of Mechanics of Solids with the consideration of "bulk masses", in terms of the stresses and strains within a bulk mass. In the literature a bulk mass is called a **continuum**.

In a continuum, we consider the more general case that at any point more than one direct stress or strain, and more than one shear stress or strain are acting simultaneously. This is called a ***multi-axial*** *stress state.*

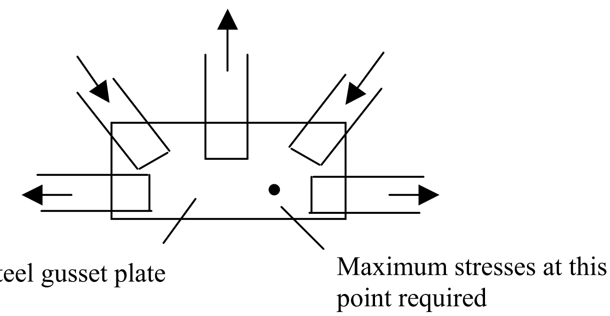


Fig. 11.1 A Gussett Plate Under Several Loads

Examine the connection in a truss shown in Fig. 11.1 above. In the practical construction of a truss, it is common for a joint to be built using a steel plate to which the members are bolted or welded. This plate is called a gusset plate. Clearly at any point in the gusset plate there will be stresses in the vertical direction and in the horizontal as well acting simultaneously. There may also be shear stresses. The critical question is - what are the maximum stresses in the plate? One may think that this can be determined by resolving the forces in each direction, dividing by the relevant area of the plate, and using the formula for a resultant (i.e. square-root-sum-of-squares). But this will not give the correct answer and the reason one may think that it would give the correct answer is because the stress is being thought of as a vector which is not so.

We learned that a force is a vector in that it is defined by both a magnitude and a direction. In our previous consideration of stress and strain we treated them as if they were vectors, but this was for convenience only since at that time our concern was with 1D components.

Recall that the direct stress is also called the "normal" stress. This hints to a very important fact about stress and strain in general (i.e. as it is conceived within a continuum). This fact is that stress or strain is not really like a vector or force, in which case there is the magnitude and direction of the force. Stress and strain are rather defined by a magnitude and 2 directions – the direction of the stress and the direction of the plane on which the stress acts. This is why the direct stress is termed the "normal" stress – it acts on a plane which is normal, or at right angles, to the direction of the stress.

This has great implications for the way engineers need to interpret how stress and strain work in the continuum - (1) when we speak of stress or strain at a point in the solid, this must mean the stresses or strains at all planes that pass through that point, called the **state of stress or strain** at that point. And (2), the state of stress or strain is relative to a coordinate system from which all planes are defined, and this enables the stress or strain on any one plane to be determined if we know the stress or strain on any other plane.

Returning to the gusset plate problem, we shall learn in the following sections that at any point in the plate, in the horizontal and vertical directions all stresses will be simultaneously present, but that there will be a plane at angle θ to the horizontal where the maximum normal stresses are located. These stresses are called the **principal stresses** and on the plane at which they act, there will be no shear stresses. There will also be another plane where the shear stresses will be a maximum, but the principal stresses will not be zero. Furthermore, every point on the plate can have different principal stresses and maximum shear stresses acting on planes at different directions (the same situation exists for the strains as well).

If the truss physically existed and we wanted to determine the principal stresses at any point on the gusset plate and their directions, the following procedure can be used:-

1. Measure the principal strains and the planes on which they act by installing a particular type of strain gauge called a rosette, at the point of interest.
2. Given the principal strains, use the constitutive relations for a continuum to calculate the principal stresses at the point.
3. If required, one can then use the transformation equations to determine the stresses or strains on any plane relative to the plane of the principal stresses or strains, in order to get the state of stress or strain at the point. These transformation equations can be solved either by matrix algebra, or graphically by the Mohr's Circle Method.

In the following sections, these items and the theories behind them are presented.

IF #56 Stress and strain are not vector quantities. They are completely defined when the direction of the plane on which the force acts is considered, in addition to the magnitude and direction of the force.

a. <u>3D Strain and Plane (2D) Strain</u>

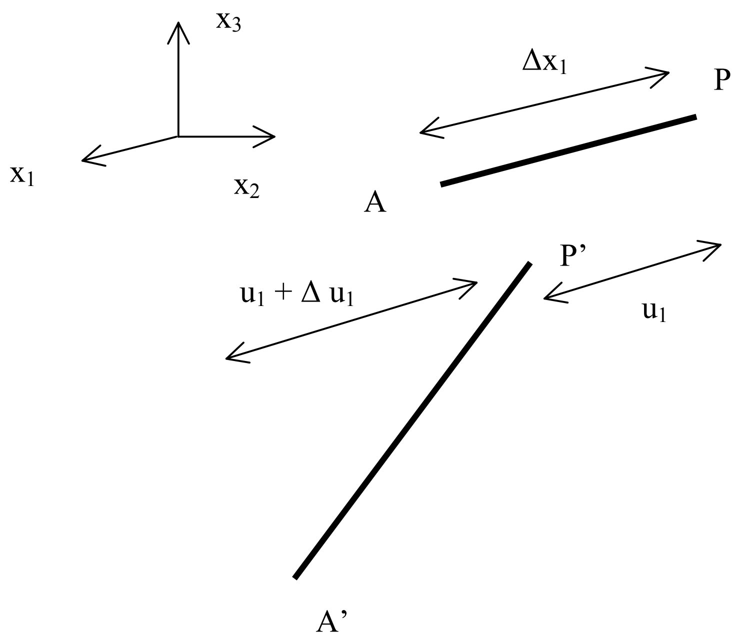


Fig. 11.2 Distortion of a Line Element PA

Consider the displacement of a particle such as P in Fig. 11.2. Its displacement can be resolved into components u_1, u_2, u_3 parallel to the coordinate axes x_1-x_2-x_3. Next consider a line element PA of length Δx_1 originally lying parallel to the x_1 axis. The displacement in the x_1 direction of the point A, accurate to the first order of a series expansion in Δx_1 is,

$$u_1 + (\partial u_1/\partial x_1)\, \Delta x_1 \qquad (11.1)$$

Notice the partial differential since the displacement of P'A' is in all 3 dimensions.

The increase in length of the element PA due to deformation is thus,

$$(\partial u_1/\partial x_1)\, \Delta x_1 \qquad (11.2)$$

As the strain is the change in length over the original length, then the strain at point P' in the x_1 direction is given by,

$$\varepsilon_{11} = \partial u_1/\partial x_1 \qquad (11.3)$$

Similar equations can be derived for line elements originally lying parallel to the x_2 and x_3 axes, so we get

$\varepsilon_{22} = \partial u_2/\partial x_2$ (11.4)

$\varepsilon_{33} = \partial u_3/\partial x_3$ (11.5)

These are called the **normal strains**.

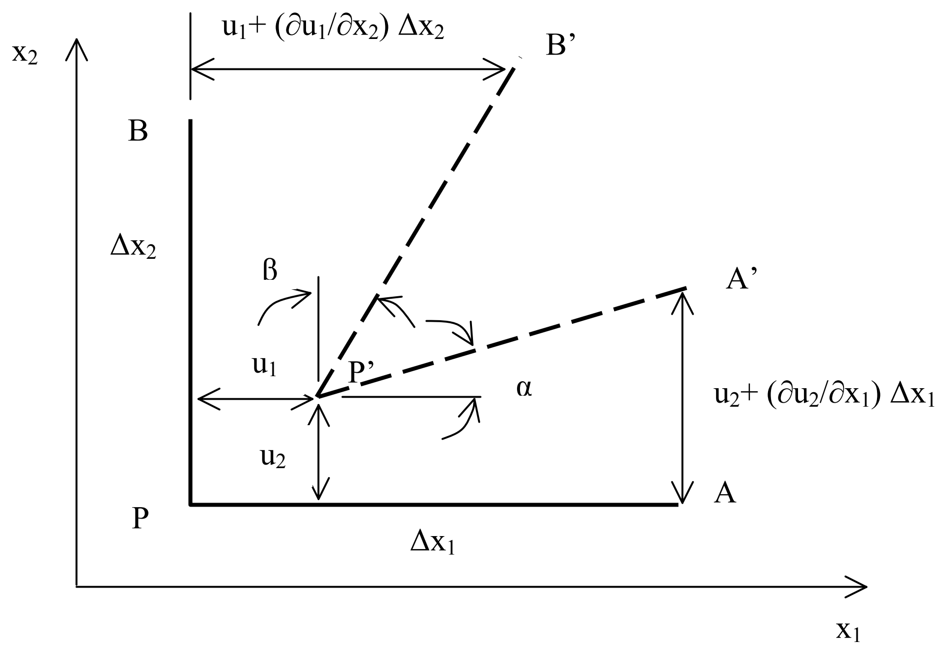


Fig. 11.3 Distortion of Line Elements PA, PB

With respect to shearing strain, consider the distortion of the original 90 degree angle between the line elements PA and PB, as shown in Fig. 11.3.

The displacement of the point A in the x_2 direction is,

$u_2 + (\partial u_2/\partial x_1)$ (11.6)

The displacement of the point B in the x_1 direction is,

$u_1 + (\partial u_1/\partial x_2)$ (11.7)

The deformed line element P'A' is inclined to the initial direction of PA by a small degree α equal to $\partial u_2/\partial x_1$. Likewise, the angle between P'B' and PB is β and is equal to $\partial u_1/\partial x_2$.

As the shearing strain is defined as the change in initial right angle on deformation, the shearing strain ε_{12} between the planes x_1-x_3 and x_2-x_3 is $\alpha + \beta$, with x_3 perpendicular to the paper . Hence,

$$\varepsilon_{12} = \partial u_1/\partial x_2 + \partial u_2/\partial x_1 \quad (11.8)$$

Similarly for the planes x_1-x_2 and x_1-x_3, and x_2-x_1 and x_2-x_3, we get

$$\varepsilon_{13} = \partial u_3/\partial x_1 + \partial u_1/\partial x_3 \quad (11.9)$$

$$\varepsilon_{23} = \partial u_3/\partial x_1 + \partial u_1/\partial x_3 \quad (11.10)$$

Equations 11.3 to 11.5, and 11.8 to 11.10, are the strain-displacement equations that relate the 6 independent components of strain, to the 3 components of displacement, given a rectangular Cartesian coordinate system.

In many problems of interest in civil engineering, suitable analysis results can be obtained if it is assumed that any strain with a component in the x_3 direction is zero. An example of this is a retaining wall in which case the x_3 direction is set in the direction of the length of the wall.

Such a state of deformation is called **plane strain**. Hence for the state of plane strain,

$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$ and

ε_{11}, ε_{22} and $\varepsilon_{12} \neq 0$.

b. 2D Strain Transformation

In the introduction to this chapter, it was stated that if a plate is subjected to forces in different directions simultaneously, the region of greatest interest to the civil engineer need not be in the direction of any of the forces. Rather this region, defined as some plane through a point in the material, will generally be at an angle θ relative to any direction chosen as the reference direction. It is therefore very important to be able to determine quantities (i.e. strains or stresses) as a function of the angle of orientation θ, of any plane at a point.

The deformation of a line element within the solid must be consistent regardless of the plane from which the deformation is being observed. Due to this geometric consistency, if we know the deformation relative to one coordinate system, it is possible to know the deformation relative to any other coordinate system at an angle θ relative to the first. The determination of one state from another is termed a transformation. Hence we are concerned in this section, with the equations of **strain transformation**.

Let us consider the state of deformation in a plane. Note that this is not the same as the state of plane strain, but enables the derivation of the strain transformation equations for a plane.

Also, let us simplify the symbols for the strains by using the contracted notation for engineering strain. Additionally, we will use the conventional symbol of γ for shear strain. Hence, ε_{11} becomes ε_x, ε_{22} becomes ε_y, and ε_{12} becomes γ_{xy}.

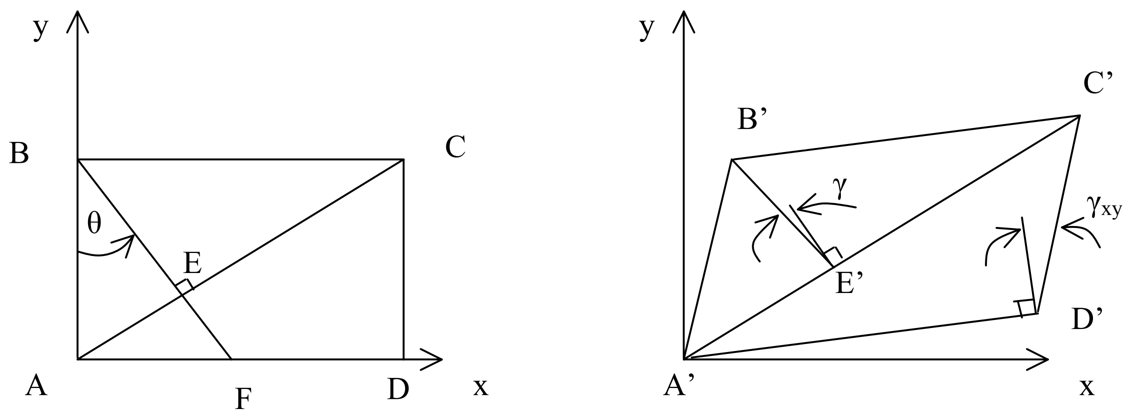
<u>Strain Normal to A Plane Oriented θ From an x-y Coordinate System</u>

Given strains ε_x, ε_y, γ_{xy} in an x-y coordinate system, what is the strain normal to a plane oriented θ from the x-y coordinate system?

The normal strain is given by,

$$\varepsilon_n = (\varepsilon_x + \varepsilon_y)/2 + [(\varepsilon_x - \varepsilon_y) \cos 2\theta] /2 + (\gamma_{xy}/2) \sin 2\theta \qquad (11.11)$$

This is derived as follows.



(a) Undeformed (b) Deformed

Fig. 11.4 Strain on a Plane ABCD

The strain normal to the plane FB is (A'C' – AC)/AC.

If δx is the increase in length from AD to A'D' and δy from CD to C'D' then

$$A'D' = AD + \delta x = AD(1 + \delta x/AD) = AD\,(1 + \varepsilon_x)$$
$$C'D' = CD + \delta Y = CD(1 + \delta y/CD) = CD\,(1 + \varepsilon_y)$$

Likewise,

$$A'C' = AC\,(1 + \varepsilon_n)$$

Now

$(A'C')^2 = (A'D')^2 + (C'D')^2 – 2A'D' . C'D' \cos(90 + \gamma_{xy})$

$(AC)^2(1 + \varepsilon_n)^2 = (AD)^2 (1+ \varepsilon_x)^2 + (CD)^2(1+ \varepsilon_y)^2 + 2AD(1+ \varepsilon_x) CD (1+ \varepsilon_y)\sin \gamma_{xy}$ (11.12)

Neglecting higher order terms, and as $\sin \gamma_{xy} \approx \gamma_{xy}$,

$(AC)^2(1 + 2\varepsilon_n) = (AD)^2 (1+ 2\varepsilon_x) + (CD)^2(1+ 2\varepsilon_y) + 2AD . CD \gamma_{xy}$ (11.13)

But $(AC)^2 = (AD)^2 + (CD)^2$ hence,

$(AC)^2 (2\varepsilon_n) = (AD)^2 (2\varepsilon_x) + (CD)^2(2\varepsilon_y) + 2AD . CD \gamma_{xy}$ (11.14)

Dividing by $2(AC)^2$,

$\varepsilon_n = \varepsilon_x \cos^2 \theta + \varepsilon_y \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta$ (11.15)

$\varepsilon_n = (\varepsilon_x + \varepsilon_y)/2 + [(\varepsilon_x – \varepsilon_y) \cos 2\theta] /2 + (\gamma_{xy}/2) \sin 2\theta$ (11.16)

Shear Strain on A Plane Oriented θ From an x-y Coordinate System

Given strains ε_x, ε_y, γ_{xy} in an x-y coordinate system, what is the shear strain on a plane oriented θ from the x-y coordinate system?

The shear strain is given by,

$\gamma_s /2 = [(\varepsilon_x - \varepsilon_y) \sin 2\theta] /2 - (\gamma_{xy}/2) \cos 2\theta$ (11.17)

This is derived as follows.

From Fig. 11.4 and as for the case of normal strain,

$A'B' = AB (1 + \varepsilon_y)$

$A'E' = AE (1 + \varepsilon_n)$

$E'B' = EB (1 + \varepsilon_{n+90})$

Where ε_{n+90} is the direct strain in a direction at 90 degrees to ε_n.

Now,

$(A'B')^2 = (A'E')^2 + (E'B')^2 – 2A'E' . E'B' \cos (90 – \gamma_s)$

$(AB)^2(1 + \varepsilon_y)^2 = (AE)^2 (1+ \varepsilon_n)^2 + (EB)^2(1+ \varepsilon_{n+90})^2 - 2AE(1+ \varepsilon_n) EB (1+ \varepsilon_{n+90}) \cos (90- \gamma_s)$

But $\cos (90- \gamma_s) = \sin \gamma_s \approx \gamma_s$ and ignoring higher order terms,

$(AB)^2(1 + 2\varepsilon_y) = (AE)^2 (1+ 2\varepsilon_n) + (EB)^2(1+ 2\varepsilon_{n+90}) + 2AE . EB \gamma_s$

But $(AB)^2 = (AE)^2 + (EB)^2$ and dividing by $2(AB)^2$ we get,

$$\varepsilon_y = \varepsilon_n \sin^2 \theta + \varepsilon_{n+90} \cos^2 \theta - \gamma_s \sin \theta \cos \theta \qquad (11.18)$$

$$½ \gamma_s \sin 2\theta = ½ (\varepsilon_{n+90} + \varepsilon_n) + ½ (\varepsilon_{n+90} - \varepsilon_n) \cos 2\theta - \varepsilon_y \qquad (11.19)$$

Now,

$\varepsilon_{n+90} = (\varepsilon_x + \varepsilon_y)/2 + [(\varepsilon_x - \varepsilon_y) \cos 2(\theta+90)] /2 + (\gamma_{xy}/2) \sin 2(\theta+90)$

$\quad = (\varepsilon_x + \varepsilon_y)/2 - [(\varepsilon_x - \varepsilon_y) \cos 2\theta] /2 - (\gamma_{xy}/2) \sin 2\theta$

Also, from (11.16), we get

$$½ (\varepsilon_{n+90} + \varepsilon_n) = ½ (\varepsilon_x + \varepsilon_y) \qquad (11.20)$$

$$½ (\varepsilon_{n+90} - \varepsilon_n) = - ½ (\varepsilon_x + \varepsilon_y) \cos 2\theta - (\gamma_{xy}/2) \sin 2\theta \qquad (11.21)$$

Substituting 11.20 and 11.21 in 11.19 and simplifying,

$$\gamma_s /2 = [(\varepsilon_x - \varepsilon_y) \sin 2\theta] /2 - (\gamma_{xy}/2) \cos 2\theta \qquad (11.22)$$

1. <u>Matrix Method of Calculation</u>

In Fig. 11.4 the line BF is the same as the y-axis, y', of another coordinate system x'-y' oriented at θ relative to the x-y coordinate system. Therefore, ε_n of equations 11.11 to 11.22 is equivalent to $\varepsilon_{x'}$ in the x'-y' system. We can therefore rewrite equation 11.15 as,

$$\varepsilon_{x'} = \varepsilon_x \cos^2 \theta + \varepsilon_y \sin^2 \theta + \gamma_{xy} \sin \theta \cos \theta \qquad (11.23)$$

Since the x'-axis is at 90 degrees to the y'-axis, we can determine $\varepsilon_{y'}$ as,

$$\varepsilon_{y'} = \varepsilon_x \sin^2 \theta + \varepsilon_y \cos^2 \theta - \gamma_{xy} \sin \theta \cos \theta \qquad (11.24)$$

For the shear strain, γ_s is equivalent to $\gamma_{x'y'}$ in the x'-y' system, and in equation 11.18, ε_{n+90} is the same as $\varepsilon_{y'}$.

Lastly, to be consistent with the same equations for the stress transformations, presented in section e, we replace the shear strain with a shear strain exactly twice its value (i.e. γ_{xy} is replaced by $2\gamma_{xy}$) and use this in equations 11.23, 11.24, and 11.18.

Then, on substituting the revised 11.23 and 11.24 in the revised 11.18 and simplifying we get,

$$\gamma_{x'y'} = -\varepsilon_x \sin\theta\cos\theta + \varepsilon_y \sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta) \quad (11.25)$$

Equations 11.23 to 11.25 can be represented in matrix format as,

$$[\varepsilon'] = [T][\varepsilon] \quad (11.26)$$

where,

$$[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

Hence to determine the strains ε' on any plane oriented θ anticlockwise from another plane with known strains ε, simply substitute in equation 11.26. The matrix multiplication can be easily done on a programmable calculator. <u>Note however that the correct value for the shear strain $\gamma_{x'y'}$ is twice the calculated value</u>.

2. <u>Mohr's Circle Graphical Method of Calculation</u>

The Mohr's Circle Method for strains has the same objective as the Matrix Method of the last section – the calculation of the strains on a plane knowing the strains on another plane. In this case, the solution is determined graphically.

Equations 11.16 and 11.22 can be rewritten as,

$$\varepsilon_n - (\varepsilon_x + \varepsilon_y)/2 = [(\varepsilon_x - \varepsilon_y)\cos 2\theta]/2 + (\gamma_{xy}/2)\sin 2\theta \quad (11.27)$$

$$\gamma_s/2 = [(\varepsilon_x - \varepsilon_y)\sin 2\theta]/2 - (\gamma_{xy}/2)\cos 2\theta \quad (11.28)$$

Squaring equations 11.27 and 11.28 and adding them together we get,

$$[\varepsilon_n - (\varepsilon_x + \varepsilon_y)/2]^2 + (\gamma_s/2)^2 = [(\varepsilon_x - \varepsilon_y)/2]^2 + (\gamma_{xy}/2)^2 \quad (11.29)$$

Equation 11.29 is the equation of a circle with radius $\frac{1}{2}\sqrt{[(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2]}$ and with the center of the circle at $[(\varepsilon_x + \varepsilon_y)/2, 0]$. When the circle is drawn, each point on the circle represents the strain values at a plane oriented 2 θ from a reference plane.

The flowing diagram is the Mohr's Circle of Strain.

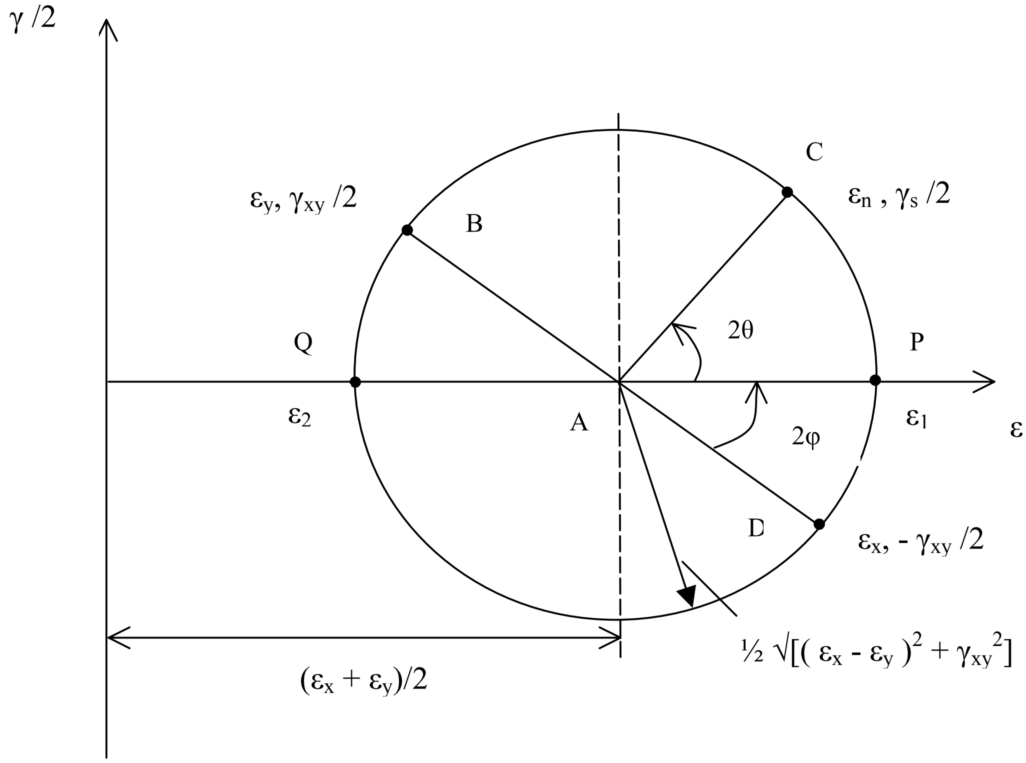
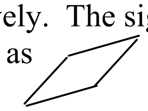
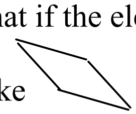


Fig. 11.5 Mohr's Circle of Strain

The procedure for drawing a Mohr's circle of strain is as follows:-

Step 1. Draw the center point A at $[(\varepsilon_x + \varepsilon_y)/2,0]$.

Step 2. Draw points D and B at $[\varepsilon_x, - \gamma_{xy} /2]$ and $[\varepsilon_y, \gamma_{xy} /2]$ respectively. The sign convention for the shear strain is that if the element deforms as  the shear strain is positive; but if like  the shear strain is negative.

Step 3. Draw a circle of diameter DB about the center.

Step 4. For any point on the circle (e.g. point C), the horizontal coordinate is the $\varepsilon_{x'}$ (i.e. ε_n) and the vertical coordinate is the $\gamma_{x'y'}/2$ (i.e. $\gamma_s/2$).

The following 2 points must be noted:-

1. The final shear strain value is to be calculated by doubling the value obtained from the Mohr's Circle.
2. $\varepsilon_{y'}$ is 180 degrees away from point $\varepsilon_{x'}$ (i.e. 180 deg away from C).

IF #57 The Mohr's Strain Circle is simply a way of determining the strains on a plane if you know the strains on another plane.

IF #58 When using the matrix method of transforming strains, the correct shear strain is twice the calculated value.

3. Principal Strains and Maximum Shear Strain

Of the infinite number of planes through a point there are certain planes where the normal strains are at the maximum value. These values are called the **principal strains** and are indicated by ε_1 and ε_2 (i.e. points P and Q) in Fig. 11.5. The principal strains act on planes oriented 2φ degrees anticlockwise and 2φ + 180 degrees anticlockwise from the planes on which ε_x and ε_y act, respectively (on the Mohr circle). ε_1 is always the numerically larger value. That is $\varepsilon_1 > \varepsilon_2$, so for principal strains of –3500 and –4000, ε_1 is the –3500. On the planes of the principal strains, the shear strain is zero.

The principal strains are given by,

$$\varepsilon_1 = (\varepsilon_x + \varepsilon_y)/2 + ½ \sqrt{[(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2]} \quad (11.30)$$

$$\varepsilon_2 = (\varepsilon_x + \varepsilon_y)/2 - ½ \sqrt{[(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2]} \quad (11.31)$$

Hence, ε_1 acts on a plane θ anticlockwise from the x-axis where,

$$\theta = ½ \tan^{-1} [\gamma_{xy}/ (\varepsilon_x - \varepsilon_y)] \quad (11.32)$$

Likewise, ε_2 acts on a plane θ anticlockwise from the x-axis where,

$$\theta = 90 + ½ \tan^{-1} [\gamma_{xy}/ (\varepsilon_x - \varepsilon_y)] \quad (11.32)$$

The maximum shear strain is given by,

$$\gamma_{s,max} = \sqrt{[(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2]} \quad (11.33)$$

Hence,

$$\gamma_{s,max} = \varepsilon_1 - \varepsilon_2 \quad (11.34)$$

Also, if equations 11.30 and 11.31 are added together we get,

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_x + \varepsilon_y \quad (11.35)$$

Equation 11.35 is called a **strain invariance** relationship since it means that the left-hand-side is always the same value regardless of the coordinate system, hence plane, in which ε_x and ε_y act.

IF #59 ε_1 is always numerically > ε_2 and the strain invariance relationship is $\varepsilon_1 + \varepsilon_2 = \varepsilon_x + \varepsilon_y$.

c. Strain Gauge Rosettes

It was stated in the introduction of this chapter that the principal strains are measured using a device called a **strain gauge rosette**. The rosette is attached to the point where one wishes to measure the principal strains.

A strain gauge rosette is actually a set of strain gauges oriented relative to each other at the same point. A strain gauge is a wire and its principle of operation is that when the material strains, the resistance of the wire changes in proportion to the strain.

However, a strain gauge can only measure strain in a direction parallel to its length so 3 gauges must work together - the rosette, to determine the principal strains. Typical arrangements are such that a constant angle of 45º, 60º, or 120º is maintained between the gauges.

Hence for a 45º rosette, the principal strains are given by,

$$\varepsilon_1 = (\varepsilon_L + \varepsilon_N)/2 + \sqrt{2}/2 \sqrt{[(\varepsilon_L - \varepsilon_M)^2 + (\varepsilon_M - \varepsilon_N)^2]} \qquad (11.36)$$

$$\varepsilon_2 = (\varepsilon_L + \varepsilon_N)/2 - \sqrt{2}/2 \sqrt{[(\varepsilon_L - \varepsilon_M)^2 + (\varepsilon_M - \varepsilon_N)^2]} \qquad (11.37)$$

where ε_L, ε_M and ε_N are the strains measured at θ, $\theta + 45º$, and $\theta + 90º$, respectively, from the plane of the principal strain, ε_1. Also,

$$\tan 2\theta = (2\varepsilon_M - \varepsilon_L - \varepsilon_N) / (\varepsilon_L - \varepsilon_N) \qquad (11.38)$$

When the principal strains and the orientations of their planes relative to the x-axis are determined using equations 11.36 to 11.38, it may still be necessary to determine ε_x and ε_y. This can be done in 2 ways:

1. By using the matrix transformation equation 11.26 and substituting a value of $-\theta$ calculated from equation 11.38.

2. By using a Mohr's Circle in terms of the principal strains, as follows.

The center of the circle has the coordinates $[1/2\ (\varepsilon_1 + \varepsilon_2),0]$, and the circle has a radius of $1/2\ (\varepsilon_1 - \varepsilon_2)$. The circle is drawn in the usual way, and the values of ε_n and γ_s, are given by the point on the circle whose radius is at 2θ anticlockwise from the normal strain axis.

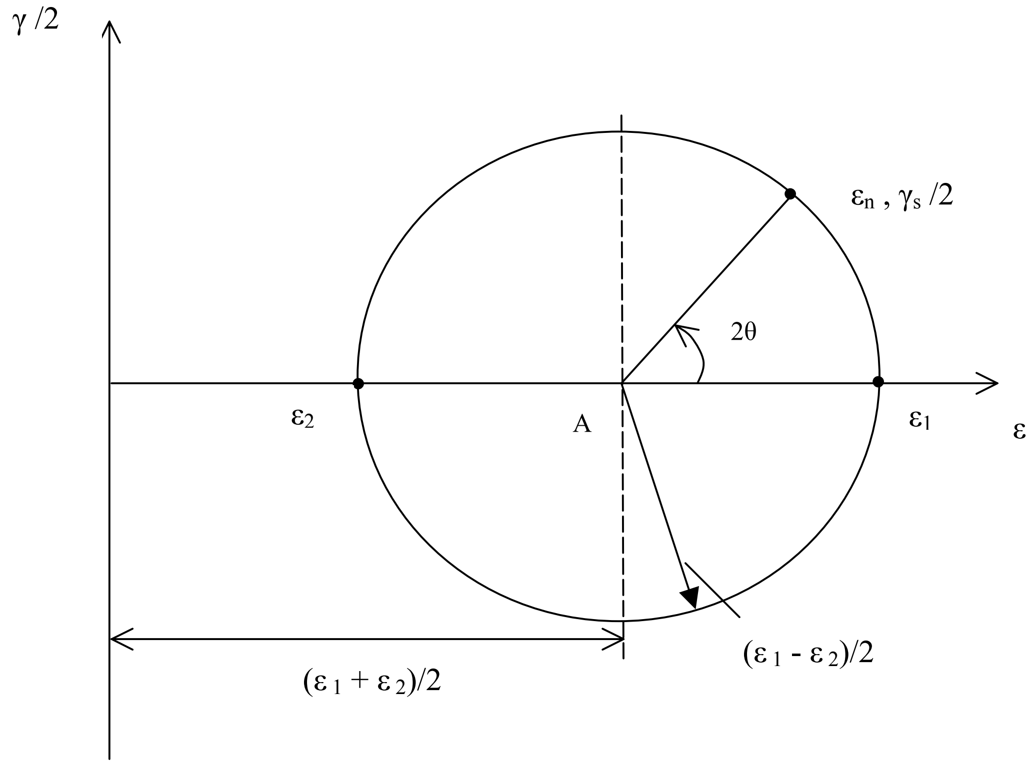


Fig. 11.6 Mohr's Circle of Principal Strains

d. Theory of 3D Stress and Plane (2D) Stress

As presented in section "a" above, strain is defined by the deformation of points in the continuum and this deformation is measurable. However in the case of stress, stress is never observed but rather inferred. Therefore a stress is a thoroughly conceptual entity required to account for the observation that different materials require different degrees of effort to achieve the same state of deformation.

We are already familiar with the concept of stress from previous chapters. However, recalling the introduction to this chapter, it was stated that the proper definition of stress requires reference to the direction of the plane on which a stress acts, in addition to the direction of the stress. Fig. 11.7 describes the theoretical stresses on an infinitesimal cube of material in a rectangular Cartesian coordinate system.

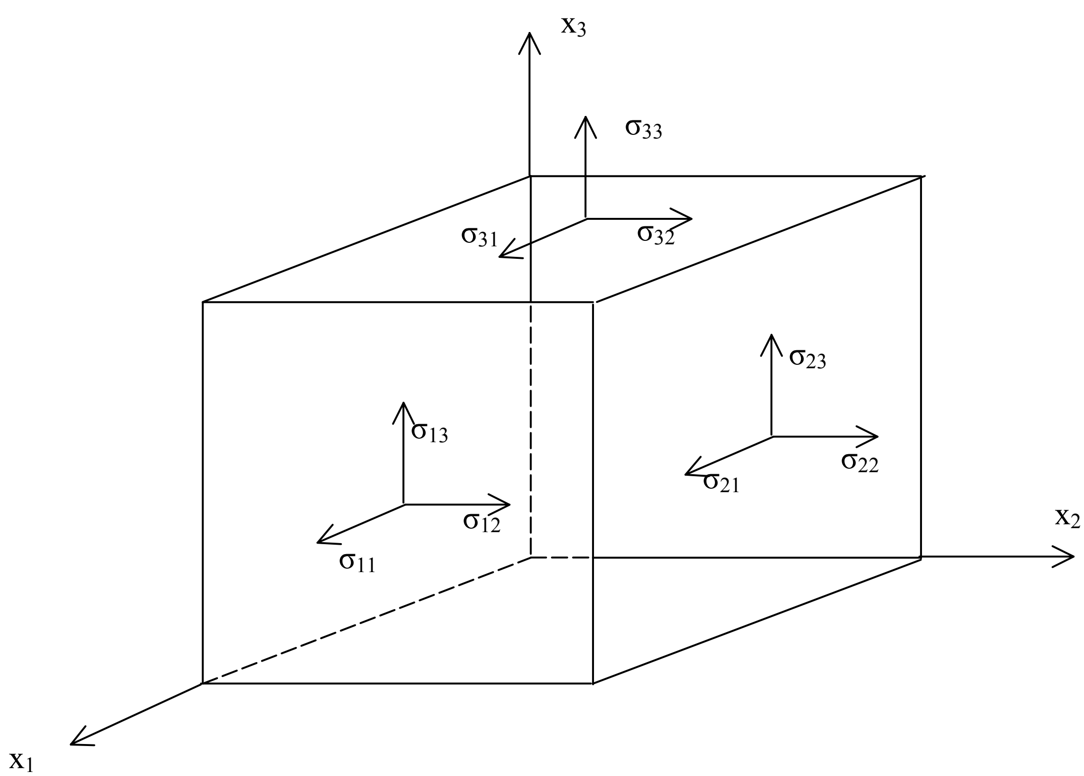
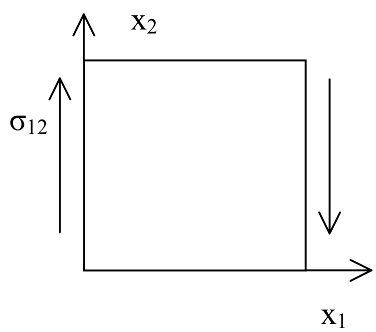


Fig. 11.7 Stresses on an Infinitesimal Volume

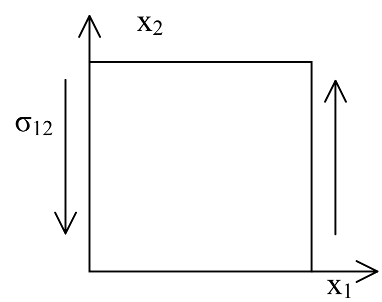
For any stress component σ_{mn} the m refers to the direction normal to axis x_m, and the n refers to the direction of the stress parallel to axis x_n.

Fig. 11.7 also indicates that in a rectangular coordinate system there are 9 components of stress – 3 normal stresses σ_{11}, σ_{22}, σ_{33} and 6 shear stresses σ_{12}, σ_{21}, σ_{13}, σ_{31}, σ_{23}, σ_{32}. Rotational equilibrium requires that each pair of the shear stresses as presented, must be equal. That is, $\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$ and $\sigma_{23} = \sigma_{32}$. Hence the σ_{21}, σ_{31} and σ_{32} are termed the **complementary shear stresses**. Therefore, the total number of independent stress components is 6.

The directions shown in Fig. 11.7 indicate the sign convention for the normal stresses. Hence tensile stresses are positive and compressive stresses are negative. With respect to shear stresses however, positive and negative shear stresses are as indicated in Fig. 11.8 below. Hence a positive shear shress causes a clockwise rotation of the element, and versa. This is also consistent with the sign convention for the shear strains as presented previously.



(a) Positive σ_{12}



(b) Negative σ_{12}

Fig. 11.8 Sign Convention for the Shear Stress

e. <u>2D Stress Transformation</u>

As for the case of the strains, it is important to be able to determine the stresses on a plane knowing the stresses on another plane orientated θ relative to the first plane. Hence in this section we are concerned with the equations of **stress transformation**.

Within the context of civil engineering, it is typically of greater practical interest to consider the 2D stress state called the state of **plane stress**. Note that this is **<u>not</u>** the same as the state of stress that corresponds to the state of plane strain previously presented.

Also, let us simplify the symbols for the stresses by using the contracted notation for engineering stress. Additionally, we will use the conventional symbol of τ for shear stress. Hence, σ_{11} becomes σ_x, σ_{22} becomes σ_y, σ_{12} becomes τ_{xy}, σ_{33} becomes σ_z, etc.

Hence for the state of plane stress,

$\sigma_z = \sigma_{xz} = \sigma_{yz} = 0$ and
σ_x, σ_y and $\tau_{xy} \neq 0$.

Note that in the case of plane stress non-zero strains still exist in the z-direction (i.e. the non-zero strains of ε_z , ε_{xz} , ε_{yz} exist). The reverse is also the case for the state of plane strain (i.e. the non-zero stresses of σ_z , σ_{xz} , σ_{yz} exist).

The stress transformation equations are derived by considering the equilibrium of a wedge obtained by cutting through an infinitesimal volume such that the inclined axis is actually the y'-axis oriented θ anticlockwise relative to the y-axis. This is shown in Fig. 11.9 below. The notation that follows is parallel to that for the strains previously presented.

<u>Stress Normal to A Plane Oriented θ From an x-y Coordinate System</u>

Given stresses σ_x, σ_y, τ_{xy} in an x-y coordinate system, what is the stress normal to a plane oriented θ from the x-y coordinate system?

The normal stress is given by,

$$\sigma_n = ½ (\sigma_x + \sigma_y) + ½ (\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy} \sin 2\theta \qquad (11.39)$$

This is derived as follows.

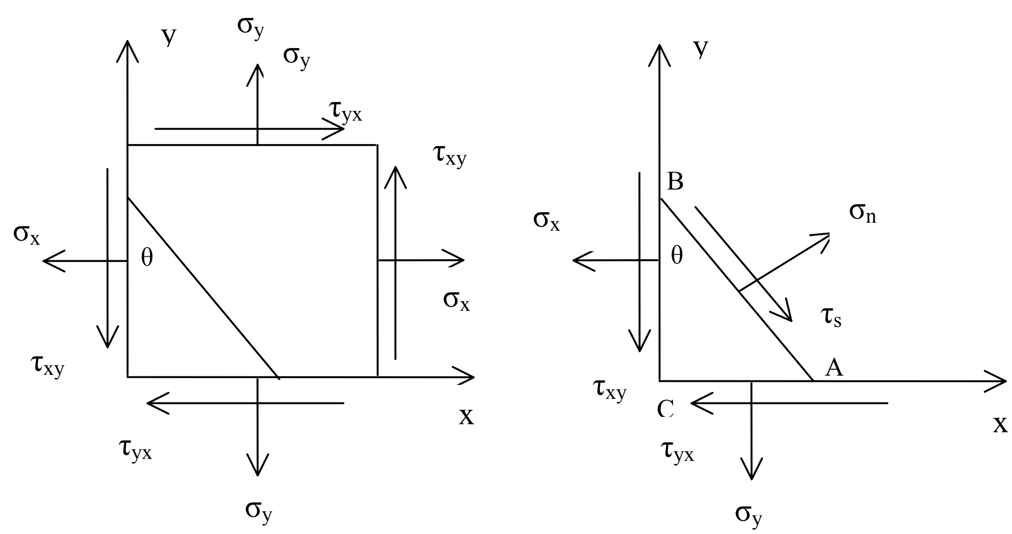


Fig. 11.9 Stresses on an Infinitesimal Wedge

Considering a unit thickness of the wedge in the direction normal to the paper, then resolving all forces normal to the inclined plane AB,

$\sigma_n AB – \sigma_x$ BC cosθ - σ_y AC sinθ - τ_{xy}BC sinθ - τ_{yx}AC cosθ = 0

Dividing by AB,

$\sigma_n – \sigma_x$ (BC/AB) cosθ - σ_y (AC/AB) sinθ = 0. But BC/AB = cosθ, and AC/AB = sinθ, so we get,

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy} \cos\theta\sin\theta + \tau_{yx} \sin\theta \cos\theta \qquad (11.40)$$

Considering that $\tau_{xy} = \tau_{yx}$ and using the trigonometrical equations, we get

$$\sigma_n = ½ (\sigma_x + \sigma_y) + ½ (\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy} \sin2\theta \qquad (11.41)$$

<u>Shear Stress on A Plane Oriented θ From an x-y Coordinate System</u>

Given stresses σ_x, σ_y, τ_{xy} in an x-y coordinate system, what is the shear stress on a plane oriented θ from the x-y coordinate system?

The shear stress is given by,

$$\tau_s = ½ (\sigma_x - \sigma_y) \sin2\theta - \tau_{xy} \cos2\theta \qquad (11.42)$$

This is derived as follows.

Now, resolving forces parallel to AB we get,

τ_s AB - σ_x BC sinθ + σ_y AC cosθ + τ_{xy}BC cosθ - τ_{yx}AC sinθ = 0

Dividing by AB,

$\tau_s - \sigma_x \cos\theta \sin\theta + \sigma_y \sin\theta \cos\theta + \tau_{yx} \cos^2\theta - \tau_{yx}\sin^2\theta = 0$

Hence,

$\tau_s = \sigma_x \cos\theta \sin\theta - \sigma_y \sin\theta \cos\theta - \tau_{xy} \cos^2\theta + \tau_{yx}\sin^2\theta$

But $\tau_{xy} = \tau_{yx}$, hence

$$\tau_s = \sigma_x \cos\theta \sin\theta - \sigma_y \sin\theta \cos\theta - \tau_{xy} \cos^2\theta + \tau_{yx}\sin^2\theta \qquad (11.43)$$

Equation 11.43 can be rewritten as,

$$\tau_s = ½ (\sigma_x - \sigma_y) \sin2\theta - \tau_{xy} \cos2\theta \qquad (11.44)$$

1. Matrix Method of Calculation

Recalling equation 11.40 and considering that σ_n is the same as σ_x' ,

$$\sigma_x' = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\cos\theta\sin\theta \qquad (11.45)$$

As σ_y' is at 90° to σ_x', equation 11.45 can be rearranged to get,

$$\sigma_y' = \sigma_x\sin^2\theta + \sigma_y\cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta \qquad (11.46)$$

Lastly, considering the sign convention for shear stress as presented in Fig. 11.8, τ_{xy} as per equation 11.43 is negative. Hence altering the signs of the terms, and noting that τ_s is the same as $\tau_{x'y'}$ we get,

$$\tau_s = -\sigma_x\cos\theta\sin\theta + \sigma_y\sin\theta\cos\theta + \tau_{xy}\cos^2\theta - \tau_{yx}\sin^2\theta \qquad (11.47)$$

Equations 11.45 to 11.47 can be put in matrix format to get the same transformation equation previously presented for strain.

$$[\sigma'] = [T][\sigma] \qquad (11.48)$$

where,

$$[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

Hence to determine the stresses σ' on any plane oriented θ anticlockwise from another plane with known stresses σ, simply substitute in equation 11.48.

Please be reminded however that if equations 11.40 and 11.45 are being used instead of the matrix equation, then the correct sign for τ_{xy} is that down-to-the-left is positive.

2. <u>Mohr's Circle Graphical Method of Calculation</u>

As in the case of the strains, a Mohr's Circle can also be developed for the stresses, as implied by equations 11.41 and 11.44. Note however that it is not necessary to half the shear stress as is the case for the shear strain.

The circle has a radius of $\frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$ and with the center of the circle at $[(\sigma_x + \sigma_y)/2, 0]$. When the circle is drawn, each point on the circle represents the stress values at a plane oriented 2θ from a reference plane.

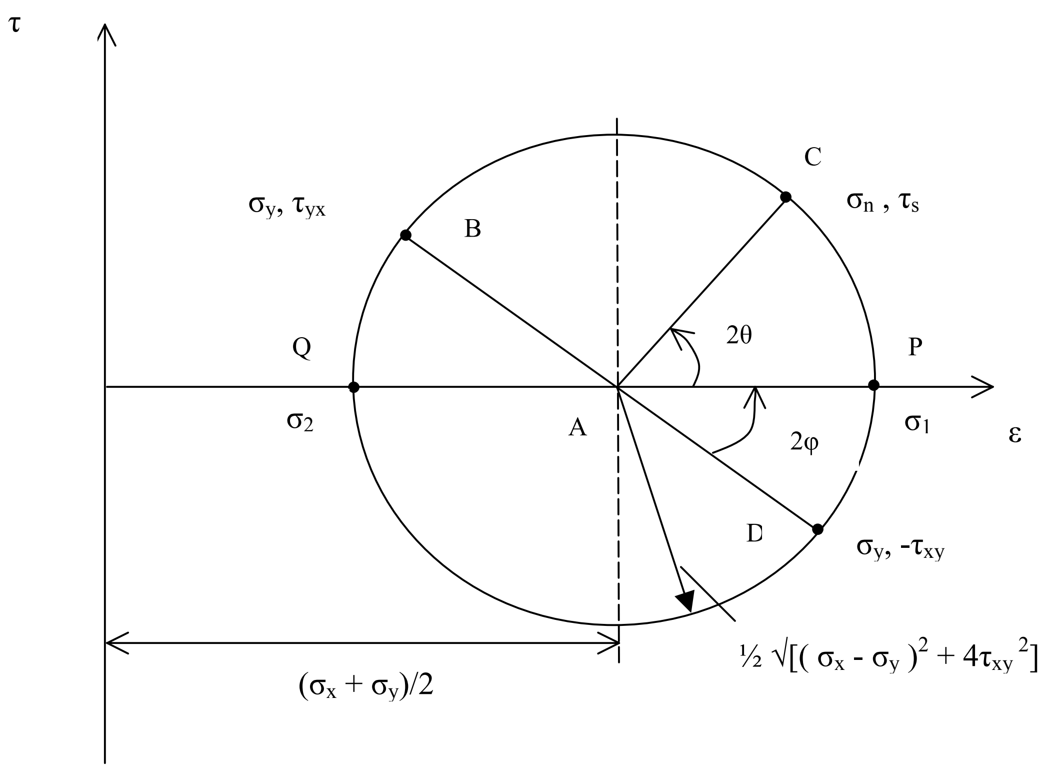


Fig. 11.10 Mohr's Circle of Stress

The procedure for drawing a Mohr's circle of stress is as follows:-

Step 1. Draw the center point A at $[(\sigma_x + \sigma_y)/2, 0]$.
Step 2. Draw points D and B at $[\sigma_y, -\tau_{xy}]$ and $[\sigma_y, \tau_{yx}]$ respectively. The sign convention for the shear stress is as indicated in Fig. 11.8.
Step 3. Draw a circle of diameter DB about the center.
Step 4. For any point on the circle (e.g. point C), the horizontal coordinate is the σ_x' (i.e. σ_n) and the vertical coordinate is the τ_{xy} (i.e. τ_s).

IF #60 The sign convention for the shear stress τ_{xy} depends on the calculation method being used. If using equation 11.43 or 44 positive is down-to-the-left or up-to-the-right. If using the matrix equation 11.48 or the Mohr's Circle method, then positive is up-to-the-right or down-to-the-left.

3. Principal Stresses and Maximum Shear Stress

The principal stresses are the maximum and minimum axial stresses that exist at a point. At the plane where the principal stresses act, the shear stress is zero on that plane. Hence from the Mohr's stress circle, as shown in Fig. 11.10, the maximum principal stress is σ_1 located at point P, and the minimum principal stress is σ_2 located at point Q. σ_1 is always the numerically larger value. That is $\sigma_1 > \sigma_2$, so for principal stresses of –35 MPa and –40 MPa, σ_1 is the –35 MPa. Therefore,

$$\sigma_1 = (\sigma_x + \sigma_y)/2 + \tfrac{1}{2} \sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}{}^2]} \quad (11.49)$$

$$\sigma_2 = (\sigma_x + \sigma_y)/2 - \tfrac{1}{2} \sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}{}^2]} \quad (11.50)$$

Adding 11.49 and 11.50 we get the stress invariance relationship:

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y \quad (11.51)$$

Also from the Mohr's stress circle, the planes on which the σ_1 and σ_2 act are φ, and φ + 90° respectively, measured anticlockwise from the x-axis.

Alternatively,

$\tan 2\varphi = \tau_{xy} / [1/2\,(\sigma_x - \sigma_y)]$ hence,

$$\varphi = \tfrac{1}{2} \tan^{-1} [2\tau_{xy} / (\sigma_x - \sigma_y)] \quad (11.52)$$

With respect to the maximum shear stress, this is represented by the top and bottom points of the Mohr's stress circle. Hence,

$$\tau_{s,max} = \tfrac{1}{2} \sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}{}^2]}$$

Note that when the shear stress is at a maximum, the corresponding normal stress is not zero but is given by, $\tfrac{1}{2}(\sigma_x + \sigma_y) = \tfrac{1}{2}(\sigma_1 + \sigma_2)$

The plane on which the maximum positive shear stress acts is given by,

$$\tan 2\theta = -(\sigma_x - \sigma_y)/(2\tau_{xy}) \quad (11.53)$$

When the principal stresses and the orientations of their planes relative to the x-axis are determined, it may still be necessary to determine σ_x and σ_y. This can be done in 2 ways:

1. By using the matrix transformation equation 11.48 and substituting a value of $-\varphi$ calculated from equation 11.52.

2. By using a Mohr's Circle in terms of the principal strains, as follows.

The center of the circle has the coordinates $[1/2\ (\sigma_1 + \sigma_2),0]$, and the circle has a radius of $1/2\ (\sigma_1 - \sigma_2)$. The circle is drawn in the usual way, and the values of σ_n and τ_s, are given by the point on the circle whose radius is at 2θ anticlockwise from the normal strain axis.

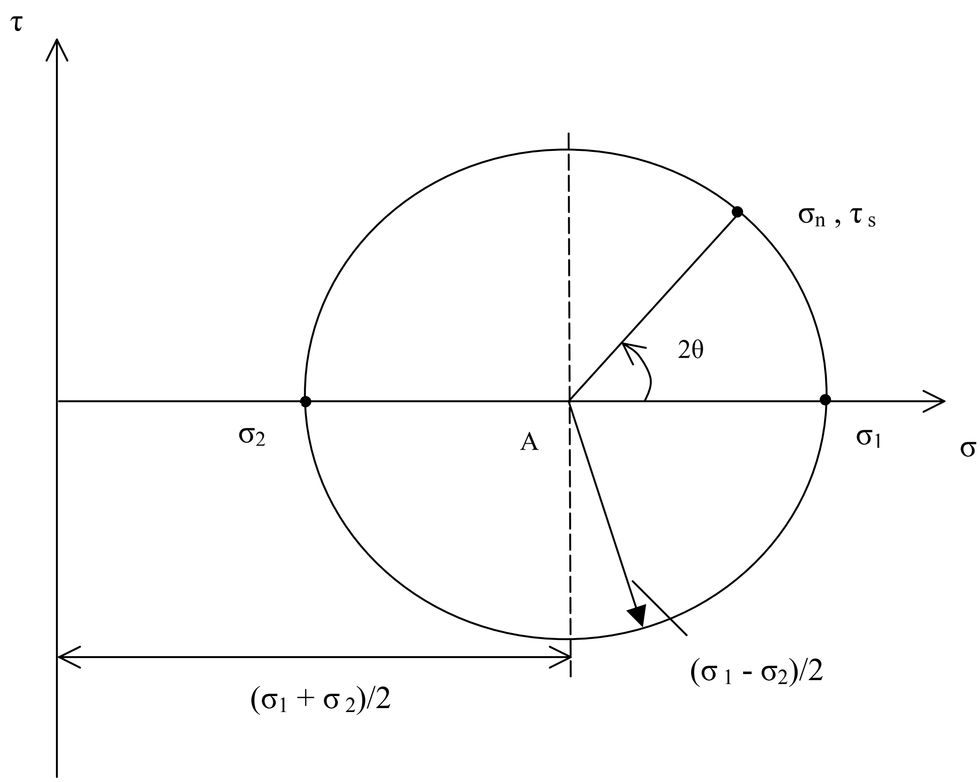


Fig. 11.11 Mohr's Circle of Principal Stresses

IF #61 σ_1 is always numerically > σ_2 and the stress invariance relationship is $\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$.

f. The Plane Stress-Strain Constitutive Relation of a Homogeneous Isotropic Material

Recall that in the introduction to this chapter a situation is discussed regarding a steel gusset plate under a state of multi-axial stress, leading to the question of the maximum stresses in the plate and their location.

In the previous sections we determined how the maximum or principal strains can be determined using strain gauge rosettes. The issue then is that given the principal strains at a point, can the principal stresses be determined at that point. The calculation of the principal stresses from the principal strains is accomplished using the constitutive relations. These are the equations that relate stress to strain and vice versa. In chapter 5 we learned the most basic constitutive relation for a uniaxial stress state as $\sigma = E\varepsilon$. We now consider the constitutive relations for the multi-axial stress state. The presentation is limited to the plane stress condition, and to materials classified as isotropic and homogeneous.

When a rod of such material is pulled in tension, or pushed in compression, there is a longitudinal (i.e. in the direction of the force) extension hence strain. Simultaneously however, the rod shrinks in a direction at right angles to the tensile force, or swells in the case of a compressive force. The direction at right angles to the axial deformation is called the **transverse** or **lateral** direction. It has been observed that the ratio of the lateral to longtitudinal strain is a constant termed the **Poisson ratio**, ν. ν is in the range 0.28 to 0.32 for most metals. Since the lateral deformation is a reduction in dimension when the longitudinal deformation is an increase, and vice versa for the case of longtitudinal compression,

Lateral strain = - ν x longtitudinal strain.

Considering this phenomenon of a lateral strain when a longtitudinal stress is applied, the strains corresponding to a biaxial tension (i.e. tension-tension) stress on an element of material is as indicated in Fig. 11.12 below.

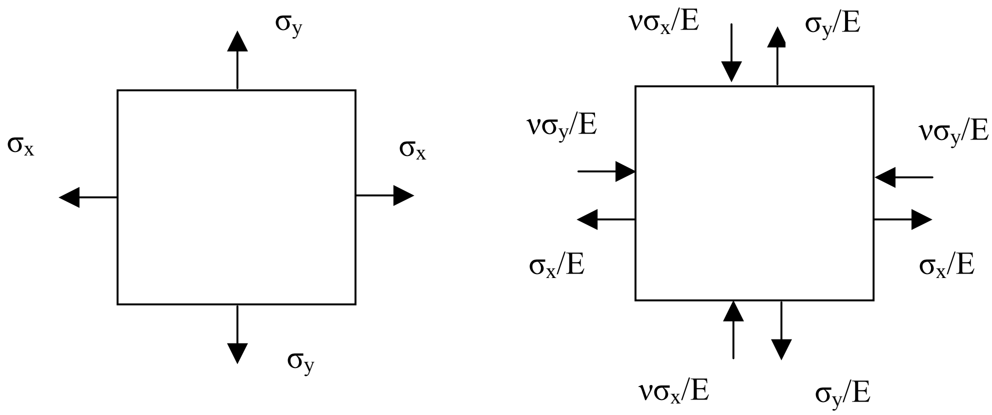


Fig. 11.12 Stresses and Corresponding Deformations

From Fig. 11.12 therefore, the total strain in the x-direction is,

$$\varepsilon_x = \sigma_x/E - \nu\sigma_x/E \quad (11.54)$$

In the y-direction,

$$\varepsilon_y = \sigma_y/E - \nu\sigma_x/E \quad (11.55)$$

We say in chapter 10 that the constitutive relation with respect to shear is,

$$\gamma_{xy} = \tau_{xy}/G \quad (11.56)$$

Equations 11.54 to 11.56 are the constitutive relations for the 2D case. In terms of principal strains they become,

$$\varepsilon_1 = \sigma_1/E - \nu\sigma_2/E \quad (11.57)$$

$$\varepsilon_2 = \sigma_2/E - \nu\sigma_1/E \quad (11.58)$$

Rearranging so that the principal stresses are the subject we get,

$$\sigma_1 = E\,(\varepsilon_1 + \nu\,\varepsilon_2)/(1 - \nu^2) \quad (11.59)$$

$$\sigma_2 = E\,(\varepsilon_2 + \nu\,\varepsilon_1)/(1 - \nu^2) \quad (11.60)$$

Equations 11.59 and 11.60 enable the calculation of the principal stresses knowing the principal strains.

Lastly, it should be mentioned that by considering the fact that shear deformation can be attained by applying normal stresses at 45°, the relationship between the various deformation constants of multi-axial deformation can be determined as,

$E = 2G(1 + \nu)$ (11.61)

g. Introduction to Failure Theory

Returning to the situation presented in the introduction, we have learned in the previous sections how to calculate the principal stresses at any point on the gusset plate by measuring the strains using a rosette and using the constitutive relations. We also learned how to determine the strains and stresses on all planes through the point.

The interest in determining the principal stresses is ultimately so that the question can be answered of whether the gusset plate in question can be safely used. The central point here is that if we believe this answer can be determined by simply comparing the principal stresses with the yield stress of the material, taking it as the limit of safety, then for the case of ductile metals, we will be wrong in vary many instances. This will be dangerous since we will have over-estimated the strength of the plate. Experimental evidence shows that for ductile metals like steel failure can occur before any of the principal stresses reach the yield stress, depending on the ratio of σ_1 to σ_2.

This has motivated the need for **failure theory** so that accurate predictions can be made of the level of stress that will cause failure for a multi-axial stress state. The following are some of the failure theories used, considering the plane stress state. In each case, the attempt is to determine, for the multi-axial stress state, the values of the principal stresses which are equivalent to the yield (i.e failure) stress when the material is placed under a simple uniaxial tension or compression test.

Maximum Principal Stress Theory

This theory is attributed to Rankine and states that failure occurs when the larger principal tensile stress, σ_1, equals the failure stress of the material in a simple tension test, or when the larger principal compressive stress, σ_2, equals the failure stress of the material in a simple compression test.

$\sigma_1 = \sigma_{y,t}$ (11.62)

$\sigma_2 = \sigma_{y,c}$ (11.63)

This is the theory referred to at the introduction of this section.

Maximum Shear Stress Theory

This theory is attributed to Tresca. It states that when failure occurs in the multi-axial stress state, the principal stresses correspond to the maximum shear stress when the material fails under simple tension. But in simple tension the maximum shear stress in the material must equal half the yield stress (see Fig. 11.11 and consider $\sigma_2 = 0$). For the 2D case this leads to,

$\sigma_1 = \sigma_{y,t}$

Note that this is the same as the maximum principal stress theory. However for the 3D case the maximum shear stress is $(\sigma_1 - \sigma_3)/2$ which would give $\sigma_1 - \sigma_3 = \sigma_{y,t}$

Maximum Shear Strain Energy Theory

This theory is called the Von Mises Theory. It states that when failure occurs in the multi-axial stress state, the principal stresses correspond to those required for the strain energy due to the shear deformation at the point, to be equivalent to the strain energy due to the shear deformation when the material fails under simple tension. For the 2D case this results in,

$$\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = \sigma_{y,t}^2 \qquad (11.64)$$

When these three failure theories are plotted on σ_1 versus σ_2 axes, the result is as shown in Fig. 11.13 if the tension and compression yield strengths are equal.

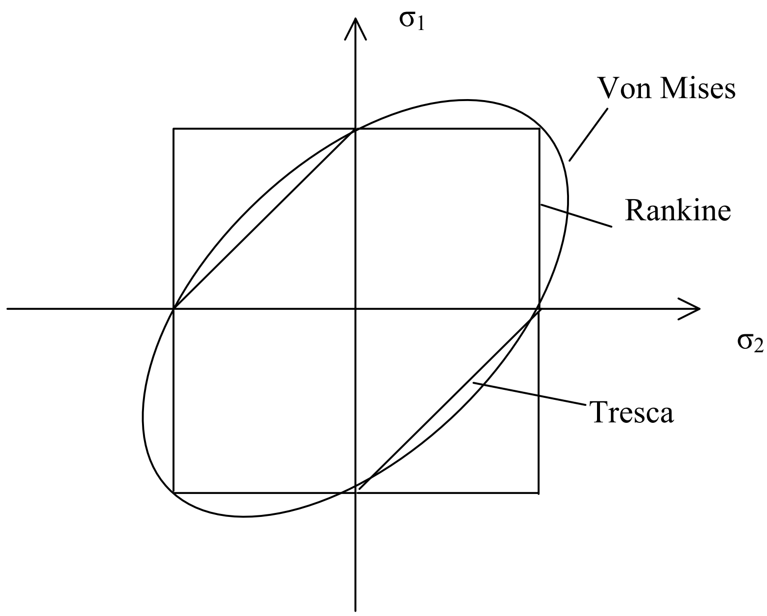


Fig. 11.13 Failure Envelopes of Some Failure Theories

A loading condition is safe if the σ_1,σ_2 coordinates fall within the envelope. Notice that the Rankine theory gives a square, the Tresca theory gives a 6-sided polygon, and the Von Mises theory gives an ellipse. However, if σ_1 and σ_2 are either both tensile or both compressive, the Tresca is the same as the Rankine.

From Fig. 11.13, if σ_1/σ_2 = -1 (i.e. tension-compression as in the case of pure torsion) as loading is increased a line progresses from the origin at a 45° angle, in the second or fourth quadrant. This line will reach the Tresca failure envelope first and the Von Mises just after that. However, the Rankine will not be reached until considerably more stress is applied. Experimental evidence indicates that the Von Mises gives the best correlation for ductile metals so this is why the Rankine theory is unsafe for such material (except when σ_1/σ_2 or σ_2/σ_1 is close to zero).

The Rankine theory results in good agreement with experimental data for brittle materials such as cast-iron, concrete and ceramics. The Tresca theory results in good agreement with experimental data for soil.